Convexity adjustments in inflation-linked derivatives with delayed payments

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Several types of inflation-linked derivatives are valued using a multi-factor version of the model of Hughston (1998) and Jarrow and Yildirim (2003). Expressions for the prices of zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments are obtained in closed forms by explicitly computing the relevant convexity adjustments. These latter results are then applied to value limited price indexation (LPI) swaps using the common factor representation methodology of Ryten (2007).

I. INTRODUCTION

In recent years, the market for inflation-linked derivatives has grown rapidly. It is fair to say that inflation is now regarded as an independent asset class. Actively-traded inflation derivatives include standard zero coupon inflation swaps, as well as more complicated products such as period-on-period inflation swaps (Mercurio 2005), inflation caps (Mercurio 2005), inflation swaptions (Kerkhof 2005), and futures contracts written on inflation (Crosby 2007).

Consider a standard zero coupon inflation swap with maturity T_M, fixed rate K, and notional amount N, which we enter into at time 0. Let X_t denote the spot CPI at time t. The payoff at time T_M of the standard zero coupon inflation swap is $N\left(X_{T_M}/X_0 - 1\right) - N\left((1+K)^{T_M} - 1\right)$. Notice that the time T_M at which the CPI is measured to specify the payout agrees with the time at which the payment takes place. While this is the common situation, often in practice the payment is delayed until some later time $T_N \geq T_M$. This delay is not just the standard two-day spot settlement lag but can be a period of a few weeks, a few months, or even several years. We will refer to such inflation swaps as "inflation swaps with delayed payments".

To see how such inflation swaps have an important economic rationale, consider a commercial property company. Suppose it has debt in the form of fixed-rate loans. It receives rents from its tenants which it wants to pay out as the inflation-linked leg of an inflation swap. It will receive fixed payments on the inflation swap which is used to pay its fixed-rate debt. Often rents will remain constant for a period of 5 years before being reviewed. They will then be revised upwards to reflect inflation over those intervening five years. So for example, suppose that the commercial property company wanted to enter into an inflation swap trade, in which it paid inflation-linked cash flows and it received fixed cash flows. The company wants to hedge the cash flows that it will receive from its tenants in years 6, 7, 8, 9 and 10. A suitable inflation swap trade would be a strip of five zero coupon inflation swaps, where the payoffs of the five zero coupon swaps are (we write only the inflation-linked leg with unit notional) as follows: At the end of year 6, the company pays $X_5/X_0 - 1$. At the end of year 7, it again pays $X_5/X_0 - 1$. Likewise, it pays $X_5/X_0 - 1$ at the end of years 8, 9 and 10.

We see that these are zero coupon inflation swaps with delayed payment, with the delay on the final strip being 5 years. Period-on-period swaps with delayed payments are also traded in the markets. We will provide formulae for both these types of inflation swap by computing the relevant convexity adjustments. Note that the issue of delayed payments should not be confused with the issue of indexation lag. Indexation lag refers to the fact that the value of the CPI in the denominator of the inflation-linked term in the payoff is, in fact, the CPI published (typically) a few weeks earlier, which, in turn, was calculated from consumer prices observed a few weeks before that. This is a different issue (although it would be possible to relate the two) and we refer the reader to Kerkhof (2005) and Li (2007).

Limited price indexation (henceforth LPI) swaps are a type of exotic inflation derivative and are very common in the United Kingdom owing to the rules by which UK pension funds are governed. We will see that the convexity adjustments required to value inflation swaps with delayed payments have a further application in the valuation of LPI swaps.

This article is structured as follows: In Section II we introduce the dynamics of nominal and real zero coupon bond prices and the spot CPI. In Section III we state the convexity adjustments required to value zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments. To our best knowledge, these results, in the context of a multi-factor Hughston and Jarrow-Yildirim model, have not appeared in the literature before, although some similar results (in the context of a two-factor Hull-White type model) are in Dodgson and Kainth (2006). These results are then applied to the valuation of limited price indexation (LPI) swaps, aided by the quasi-analytic methodology of Ryten (2007). A number of examples and comparisons are given in Section IV. We finish with a brief concluding remark in Section V. The appendix contains proofs of the convexity adjustment formulae as well as explicit formulae for the valuation of zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments.

II. MODELS FOR BOND PRICES AND THE SPOT CPI

We model the market with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$ generated by a multi-dimensional Brownian motion. The probability measure $\mathbb{Q}$ denotes the risk-neutral measure, and market prices and other information-providing processes are adapted to $\{\mathcal{F}_t\}$. Throughout the paper we assume the absence of arbitrage and the existence of a pricing kernel—these conditions ensure the existence of a unique pricing measure $\mathbb{Q}$. We let $\mathbb{E}_t[-]$ denote the expectation in $\mathbb{Q}$ conditional on $\{\mathcal{F}_t\}$.

We denote calendar time by t; time $t = 0$ will denote the initial time. Let $\{r^N_t\}$ and $\{r^R_t\}$ denote, respectively, the (continuously compounded) risk-free nominal and real short rate processes. Let $\{P^N_{tT}\}$ and $\{P^R_{tT}\}$ denote, respectively, the price process of a nominal and real zero coupon bond maturing at T. The spot CPI at time t is denoted by X_t.

A key observation for pricing inflation derivatives is that, for any times t and T_M, $t \leq T_M$, we have (Hughston 1998):

$$X_t P^R_{tT_M} = \mathbb{E}_t\left[X_{T_M}\exp\left(-\int_t^{T_M} r^N_s \mathrm{d}s\right)\right]. \tag{1}$$

This follows from the fact that the right side of (1) is the price at time t of an index-linked bond, which pays the amount X_{T_M} at time T_M. Dividing it by X_t we obtain the value in real terms of a bond that pays one unit of goods and services at time T_M. Mercurio (2005) uses this relation to value standard zero coupon inflation swaps, and shows how, given the fixed rates quoted in the markets for these swaps, the term structure of real discount factors can be obtained.

Now, we introduce the models for the dynamical equations satisfied by nominal zero coupon bond prices, real zero coupon bond prices, and the spot CPI, within the multi-factor version of the Hughston and Jarrow-Yildirim model. These are given by:

$$\frac{\mathrm{d}P^N_{tT}}{P^N_{tT}} = r^N_t \mathrm{d}t + \sum_{k=1}^{K_N} \sigma^N_{ktT}\, \mathrm{d}z^N_{kt}, \tag{2}$$

$$\frac{\mathrm{d}P^R_{tT}}{P^R_{tT}} = \left(r^R_t - \sum_{k=1}^{K_R} \rho^{RX}_k \sigma^X_t \sigma^R_{ktT} \right) \mathrm{d}t + \sum_{k=1}^{K_R} \sigma^R_{ktT}\, \mathrm{d}z^R_{kt}, \tag{3}$$

and

$$\frac{\mathrm{d}X_t}{X_t} = \left(r^N_t - r^R_t \right) \mathrm{d}t + \sigma^X_t \mathrm{d}z^X_t. \tag{4}$$

Here K_N and K_R are the number of Brownian motions driving nominal and real zero coupon bond prices respectively, $\{\mathrm{d}z^N_{kt}\}_{k=1,\ldots,K_N}$, $\{\mathrm{d}z^R_{kt}\}_{k=1,\ldots,K_R}$, and $\{\mathrm{d}z^X_t\}$ denote standard $\mathbb{Q}$-Brownian increments. Furthermore, $\{\sigma^N_{ktT}\}_{k=1,\ldots,K_N}$ and $\{\sigma^R_{ktT}\}_{k=1,\ldots,K_R}$ are volatility terms, which are assumed to be deterministic, satisfying $\sigma^N_{kTT} = 0$, and $\{\sigma^X_t\}$ is the spot CPI volatility which we also assume to be deterministic. We denote correlations (all assumed constant) by ρ with appropriate subscripts: $\mathrm{Corr}(\mathrm{d}z^N_{jt}, \mathrm{d}z^N_{kt}) = \rho^{NN}_{jk}\mathrm{d}t$, $\mathrm{Corr}(\mathrm{d}z^R_{jt}, \mathrm{d}z^R_{kt}) = \rho^{RR}_{jk}\mathrm{d}t$, $\mathrm{Corr}(\mathrm{d}z^X_t, \mathrm{d}z^N_{kt}) = \rho^{NX}_k\mathrm{d}t$, $\mathrm{Corr}(\mathrm{d}z^X_t, \mathrm{d}z^R_{jt}) = \rho^{RX}_j\mathrm{d}t$, and $\mathrm{Corr}(\mathrm{d}z^N_{jt}, \mathrm{d}z^R_{kt}) = \rho^{NR}_{jk}\mathrm{d}t$.

III. LIMITED PRICE INDEXATION (LPI) SWAPS

In this section, we will provide a valuation formula for LPI swaps. Before discussing LPI swaps, we state two preliminary propositions, the proofs of which are in the Appendix A. We will use them in this section to value LPI swaps. However, as we show in the Appendix B, they can also be used to value zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments.

Proposition 1 *Given the assumptions of Section II, for any times* t *and* T_N, $0 \le t \le T_M \le T_N$, *the following relation holds:*

$$\mathbb{E}_t \left[X_{T_M} \exp\left(-\int_t^{T_N} r^N_s \mathrm{d}s \right) \right] = X_t P^R_{tT_M} \frac{P^N_{tT_N}}{P^N_{tT_M}} \exp\left(\int_t^{T_M} C_s(T_M, T_N) \mathrm{d}s \right), \tag{5}$$

where

$$\begin{aligned} C_s(T_M, T_N) \; = \; & \sum_{k=1}^{K_N} \left(\sigma^N_{ksT_N} - \sigma^N_{ksT_M} \right) \left(\sum_{j=1}^{K_R} \rho^{NR}_{kj} \sigma^R_{jsT_M} - \sum_{j=1}^{K_N} \rho^{NN}_{kj} \sigma^N_{jsT_M} \right) \\ & + \sum_{k=1}^{K_N} \left(\sigma^N_{ksT_N} - \sigma^N_{ksT_M} \right) \rho^{NX}_k \sigma^X_s. \end{aligned} \tag{6}$$

We remark that when $T_M = T_N$ it is straightforward to verify that $C_s(T_M, T_N) = 0$, in which case equation (5) agrees with equation (1).

Proposition 2 *Given the assumptions of Section II, we have, for* $0 \le t < T_{i-1} < T_i \le T_{N_i}$,

$$\begin{aligned}
&\mathbb{E}_t\left[\frac{X_{T_i}}{X_{T_{i-1}}}\exp\left(-\int_t^{T_{N_i}} r_s^N \mathrm{d}s\right)\right] \qquad (7)\\
&= P^N_{tT_{i-1}}\frac{P^N_{tT_{N_i}}}{P^N_{tT_i}}\frac{P^R_{tT_i}}{P^R_{tT_{i-1}}}\exp\left(\int_{T_{i-1}}^{T_i} C_s(T_i, T_{N_i})\mathrm{d}s + \int_t^{T_{i-1}}\left[A_s(T_{i-1}, T_i) + B_s(T_{i-1}, T_i, T_{N_i})\right]\mathrm{d}s\right),
\end{aligned}$$

where $C_s(T_i, T_{N_i})$ *is given by (6) and where*

$$\begin{aligned}
A_s(T_{i-1}, T_i) \;=\; & \sum_{j=1}^{K_R}\left(\sigma^R_{jsT_i} - \sigma^R_{jsT_{i-1}}\right)\left(\sum_{k=1}^{K_N}\rho^{NR}_{kj}\sigma^N_{ksT_{i-1}} - \sum_{k=1}^{K_R}\rho^{RR}_{kj}\sigma^R_{ksT_{i-1}}\right)\\
& + \sum_{k=1}^{K_R}\left(\sigma^R_{ksT_{i-1}} - \sigma^R_{ksT_i}\right)\rho^{RX}_k\sigma^X_s, \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
B_s(T_{i-1}, T_i, T_{N_i}) \;=\; & \sum_{k=1}^{K_N}\sum_{j=1}^{K_N}\rho^{NN}_{kj}\left(\sigma^N_{ksT_{i-1}} - \sigma^N_{ksT_i}\right)\left(\sigma^N_{jsT_{N_i}} - \sigma^N_{jsT_i}\right)\\
& + \sum_{k=1}^{K_N}\sum_{j=1}^{K_R}\rho^{NR}_{kj}\left(\sigma^R_{jsT_i} - \sigma^R_{jsT_{i-1}}\right)\left(\sigma^N_{ksT_{N_i}} - \sigma^N_{ksT_i}\right). \qquad (9)
\end{aligned}$$

We now proceed to the valuation of LPI swaps.

Suppose that today, at time 0, we enter into an LPI swap. The LPI swap is defined via a set of fixed dates $T_0 < T_1 < T_2 < \cdots < T_{M-1} < T_M$, where $T_0 = 0$. The payment of the payoff of the swap occurs at time T^*, where $T^* = T_M$. The payoff of the inflation-linked leg of the swap at time T^* is given by

$$\prod_{i=1}^{M}\min\left(\max\left(\frac{X_{T_i}}{X_{T_{i-1}}}, 1+F\right), 1+C\right),$$

where C and F are constants with $C \ge F$. In practice, F is often zero but we will assume in the following that C and F can take on any values (positive, negative or zero) provided that $C \ge F$. We see that the role of the constants C and F is to cap and floor the period-on-period inflation rate over each period.

We remark that when $C = \infty$ and $F = -\infty$ the product 'telescopes' and the LPI swap has the same payoff as a zero coupon inflation swap. However, when C and F are finite and when $M > 1$, we need to price a swap whose payoff is path-dependent. For typical values of M (between 5 and 40, say), the only feasible methodology to price LPI swaps is by Monte Carlo simulation, but this is CPU intensive. Hence it would be desirable to have a fast, even if approximate, quasi-analytic methodology to price them. Such a methodology, based on the idea of common factor representation, is proposed in Ryten (2007). Note, however, that Ryten's model setup is rather different from ours. We will apply Ryten's idea, in order to value LPI swaps, within the setup of our multi-factor version of the model of Hughston (1998) and Jarrow and Yildirim (2003).

Let us begin by introducing some additional notation. We let $\mathbb{Q}_{T^*}$ denote the probability measure defined with respect to the numeraire which is the zero coupon bond maturing at time T^*. Similarly, we let $\mathbb{E}_t^{T^*}[-]$ denote the expectation with respect to the measure $\mathbb{Q}_{T^*}$ conditional on $\{\mathcal{F}_t\}$. Suppose that we have a T_M year LPI swap with M periods. Let X_i denote $X_{T_i}/X_{T_{i-1}}$ for $i = 1, 2, \ldots, M$. In Li (2007), it is shown that $\ln X_i$ for each $i = 1, 2, \ldots, M$ is normally distributed in our model, and that we can calculate the covariance matrix $\mathrm{cov}(\ln X_i, \ln X_j)$ for each i, j. In general, none of the elements of this covariance matrix vanish because $\ln X_i$ is not independent of $\ln X_j$ for any i, j. This lack of independence complicates the problem of pricing an LPI swap. The idea of Ryten (see also Jackel 2004) is to replace the covariance matrix $\mathrm{cov}(\ln X_i, \ln X_j)$ for each i, j by another matrix, which is close to the actual correlation matrix in some sense, but in which the off-diagonal elements have a simple structure. This is achieved by generating all the co-dependence between $\ln X_i$ and $\ln X_j$ through a single common factor (in fact, Ryten also considers the case of two common factors but we will, for the sake of brevity, only consider one).

We remark that it is easy to show (Li 2007) that $\ln X_i = \ln X_{T_i}/\ln X_{T_{i-1}}$, for each $i = 1, 2, \ldots, M$, is distributed as multi-variate normal random variables in the measure $\mathbb{Q}_{T^*}$. That is to say, $\ln X_i$ is Gaussian with deterministic drift and volatility under $\mathbb{Q}_{T^*}$. Hence we can write X_i in the form $X_i = \exp\left(a_i z_i + b_i\right)$, where $z_i \sim N(0, 1)$; $\mathrm{cov}(\ln X_i, \ln X_j) = \mathrm{cov}(z_i, z_j)a_i a_j$; and $\mathbb{E}_t\left[X_i\right] = \exp\left(b_i + \frac{1}{2}a_i^2\right)$.

The key idea of Ryten (2007) is to replace X_i by $\hat{X}_i$ defined via

$$\hat{X}_i \equiv \exp\left[b_i + a_i\left(\hat{a}_i w + \sqrt{(1-\hat{a}_i^2)}\varepsilon_i\right)\right],$$

where the system $\{w, \varepsilon_1, \ldots, \varepsilon_M\}$ is a family of independent $N(0, 1)$ variates. The variates $\hat{X}_1, \ldots, \hat{X}_M$ represent the variates $X_1, \ldots, X_M$ via one common factor w and additional individual idiosyncratic random variables $\{\varepsilon_i\}_{i=1,2,\ldots,M}$. Note that the common factor w is an abstract factor and does not necessarily correspond to any market-observable.

From Ryten (2007), which in turn references Jackel (2004), we know that when $M \geq 3$ we can approximate $\hat{a}_k$ by

$$\hat{a}_k \approx \exp\left[\frac{1}{M-2}\left(\bar{k}_k - \frac{\sum_{i=1}^{M}\bar{k}_i}{2\,(M-1)}\right)\right],$$

where $\bar{k}_k = \sum_{i\neq k}^{M} \ln\left[\mathrm{cov}(\ln X_i, \ln X_k)\right]$, $k = 1, 2, \ldots, M$. In the cases for which $M = 1$ or $M = 2$, we do not need an approximation. Indeed, if $M = 1$ then we have (trivially) $\hat{a}_1 = 1$; likewise if $M = 2$, then we have (from Cholesky decomposition) $\hat{a}_1 = 1$ and $\hat{a}_2 = \mathrm{Corr}\left(\ln X_1, \ln X_2\right)$.

Note that the relations $\mathbb{E}_0^{T^*}[\hat{X}_i] = \mathbb{E}_0^{T^*}\left[X_i\right]$ and $\mathrm{var}[\ln \hat{X}_i] = \mathrm{var}\left[\ln X_i\right]$ are valid for all $i = 1, 2, \ldots, M$ and for all value of M. However, if $M \geq 3$, then $\mathrm{cov}(\hat{X}_i, \hat{X}_j)$ is only an approximation to $\mathrm{cov}(X_i, X_j)$ when $i \neq j$.

We now apply Ryten's idea in order to value LPI swaps. By changing the measure to $\mathbb{Q}_{T^*}$ and using Girsanov's theorem, the price at time $T_0 = 0$ of the inflation-linked leg of the

LPI swap is:

$$
\begin{aligned}
&\mathbb{E}_0\left[\exp\left(-\int_0^{T^*} r_s^N \mathrm{d}s\right)\prod_{i=1}^{M}\min\left(\max\left(\frac{X_{T_i}}{X_{T_{i-1}}},1+F\right),1+C\right)\right]\\
&= P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\frac{X_{T_i}}{X_{T_{i-1}}},1+F\right),1+C\right)\right]\\
&\approx P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\hat{X}_i,1+F\right),1+C\right)\right]\\
&= P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\min\left(\max\left(\hat{X}_i,1+F\right),1+C\right)\middle| w\right]\right]\\
&= P_{0T^*}^N\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M}\mathbb{E}_0^{T^*}\left[\min\left(\max\left(\hat{X}_i,1+F\right),1+C\right)\middle| w\right]\right] \qquad (10)
\end{aligned}
$$

By assumption the random variables ε_i are independent, and consequently, conditional on w, the variates $\hat{X}_i$ are also independent, i.e. $\mathrm{cov}\left(\hat{X}_i,\hat{X}_j\middle| w\right)=0$, when $i\neq j$. Therefore, we see that the conditional expectation of the product in the last but one line of equation (10) becomes a product of conditional expectations in the last line. We have used $\approx$ (approximately equals) in the third line of equation (10) because the variates $\hat{X}_i$ are, in general (i.e. when $M\geq 3$), only an approximate representation of the variates X_i for $i=1,2,\ldots,M$.

In order to evaluate equation (10) we need to compute the $\mathbb{Q}_{T^*}$-expectation of X_i and the covariance matrix $\mathrm{cov}(\ln X_i,\ln X_j)$. The latter is shown in Li (2007) to be given by

$$
\begin{aligned}
\mathrm{cov}\left(\ln X_i,\ln X_j\right) &= \int_0^{T_{i-1}}\mathrm{cov}\Big(\sum_{k=1}^{K_R}(\sigma_{ksT_i}^R-\sigma_{ksT_{i-1}}^R)dz_{ks}^R-\sum_{p=1}^{K_N}(\sigma_{psT_i}^N-\sigma_{psT_{i-1}}^N)dz_{ps}^N,\\
&\qquad \sum_{k=1}^{K_R}(\sigma_{ksT_j}^R-\sigma_{ksT_{j-1}}^R)dz_{ks}^R-\sum_{p=1}^{K_N}(\sigma_{psT_j}^N-\sigma_{psT_{j-1}}^N)dz_{ps}^N\Big)\mathrm{d}s\\
&+\int_{T_{i-1}}^{T_i}\mathrm{cov}\Big(\sigma_s^X dz_s^X+\sum_{k=1}^{K_R}\sigma_{ksT_i}^R dz_{ks}^R-\sum_{p=1}^{K_N}\sigma_{psT_i}^N dz_{ps}^N,\\
&\qquad \sum_{k=1}^{K_R}(\sigma_{ksT_j}^R-\sigma_{ksT_{j-1}}^R)dz_{ks}^R-\sum_{p=1}^{K_N}(\sigma_{psT_j}^N-\sigma_{psT_{j-1}}^N)dz_{ps}^N\Big)\mathrm{d}s
\end{aligned}
$$

when $j>i$, whereas when $j=i$ we have

$$
\begin{aligned}
\mathrm{var}\left(\ln X_i\right) \equiv \sigma_{\ln X_i}^2 &= \int_0^{T_{i-1}}\mathrm{var}\left(\sum_{k=1}^{K_R}(\sigma_{ksT_i}^R-\sigma_{ksT_{i-1}}^R)dz_{ks}^R-\sum_{p=1}^{K_N}(\sigma_{psT_i}^N-\sigma_{psT_{i-1}}^N)dz_{ps}^N\right)\mathrm{d}s\\
&+\int_{T_{i-1}}^{T_i}\mathrm{var}\left(\sigma_s^X dz_s^X+\sum_{k=1}^{K_R}\sigma_{ksT_i}^R dz_{ks}^R-\sum_{p=1}^{K_N}\sigma_{psT_i}^N dz_{ps}^N\right)\mathrm{d}s.
\end{aligned}
$$

The former can also be computed since it follows from the Girsanov theorem that the $\mathbb{Q}_{T^*}$-expectation of X_i is

$$\mathbb{E}_0^{T^*}\left[\frac{X_{T_i}}{X_{T_{i-1}}}\right] = \frac{1}{P_{0T^*}^N}\,\mathbb{E}_0\left[\exp\left(-\int_0^{T^*} r_s^N \mathrm{d}s\right)\frac{X_{T_i}}{X_{T_{i-1}}}\right]. \tag{11}$$

The $\mathbb{Q}_{T^*}$-expectation of X_i can then be evaluated explicitly by use of Propositions 1 and 2. Specifically, when $i = 1$, we find, since $T_0 = 0$, that (11) implies

$$\mathbb{E}_0^{T^*}[X_i] = \frac{P_{0T_1}^R}{P_{0T_1}^N}\exp\left(\int_0^{T_1} C_s(T_1, T^*)\mathrm{d}s\right), \tag{12}$$

whereas when $i > 1$ we obtain

$$\begin{aligned}\mathbb{E}_0^{T^*}[X_i] \;=\; & \frac{P_{0T_{i-1}}^N}{P_{0T_i}^N}\frac{P_{0T_i}^R}{P_{0T_{i-1}}^R}\exp\left(\int_{T_{i-1}}^{T_i} C_s(T_i, T^*)\mathrm{d}s\right.\\ & \left. + \int_0^{T_{i-1}} \left[A_s(T_{i-1}, T_i) + B_s(T_{i-1}, T_i, T^*)\right]\mathrm{d}s\right).\end{aligned} \tag{13}$$

Furthermore, since X_i is lognormal, we can use the standard result that if we denote by $\mu_{\ln X_i}$ and $\sigma^2_{\ln X_i}$ the mean and variance of $\ln X_i$, then $\mathbb{E}_0^{T^*}[X_i] = \exp\left(\mu_{\ln X_i} + \frac{1}{2}\sigma^2_{\ln X_i}\right)$ for $i = 1, 2, \ldots, M$. Hence we obtain the expectation of $\ln X_i$: $\mu_{\ln X_i} = \ln\left(\mathbb{E}_0^{T^*}[X_i]\right) - \frac{1}{2}\sigma^2_{\ln X_i}$.

Now we can use the following well-known result: *If* $X \sim N(\mu_X, \sigma_X^2)$, $W \sim N(0,1)$, *and* ρ_{XW} *is the correlation between* X *and* W, *then* $X \mid (W = w)$ *is normally distributed and, furthermore,* $\mathbb{E}[X \mid W = w] = \mu_X + \rho_{XW}\sigma_X w$ *and* $\mathrm{var}[X \mid W = w] = \sigma_X^2(1 - \rho_{XW}^2)$.

We can calculate the correlation between $\ln \hat{X}_i$ and the common factor w. Indeed, since $\ln \hat{X}_i$ is normally distributed with variance a_i^2, and since

$$\mathrm{cov}\left(\ln \hat{X}_i, w\right) = \mathrm{cov}\left(a_i\left(\hat{a}_i w + \sqrt{(1-\hat{a}_i^2)}\varepsilon_i\right), w\right) = a_i\hat{a}_i,$$

we deduce that the correlation between $\ln \hat{X}_i$ and w is $\hat{a}_i$ for each $i = 1, 2, \ldots, M$. Now we recall that $\mathbb{E}_0^{T^*}[\ln \hat{X}_i] = \mathbb{E}_0^{T^*}[\ln X_i] = \mu_{\ln X_i}$ and that $\mathrm{var}[\ln \hat{X}_i] = \mathrm{var}[\ln X_i] = \sigma^2_{\ln X_i}$. Then using the result above we get

$$\mathbb{E}_0^{T^*}\left[\ln \hat{X}_i | w\right] = \mu_{\ln X_i} + \hat{a}_i \sigma_{\ln X_i} w,$$

$$\bar{\sigma}_i^2 \equiv \mathrm{var}\left[\ln \hat{X}_i | w\right] = \sigma^2_{\ln X_i}\left(1 - \hat{a}_i^2\right),$$

and

$$\bar{F}_i \equiv \mathbb{E}_0^{T^*}[\hat{X}_i \mid w] = \exp\left(\mu_{\ln X_i} + \hat{a}_i\,\sigma_{\ln X_i} w + \tfrac{1}{2}\bar{\sigma}_i^2\right)$$

for $i = 1, 2, \ldots, M$.

Finally equation (10) becomes:

$$P_{0T^*}^N \mathbb{E}_0^{T^*}\left[\prod_{i=1}^M \left(\bar{F}_i - Call\left(\bar{F}_i, 1 + C, \bar{\sigma}_i^2\right) + Put\left(\bar{F}_i, 1 + F, \bar{\sigma}_i^2\right)\right)\right], \tag{14}$$

where $Call\left(\bar{F}_i, 1+C, \bar{\sigma}_i^2\right)$ and $Put\left(\bar{F}_i, 1+F, \bar{\sigma}_i^2\right)$ are, respectively, the undiscounted prices of a call option with strike $1+C$ and a put option with strike $1+F$, in the Black (1976) formula, when the forward price is $\bar{F}_i$ and the integrated variance is $\bar{\sigma}_i^2$. Note that each term in the product in equation (14) depends on the common factor w through $\bar{F}_i$ and $\bar{\sigma}_i^2$, and w has a standard normal $N(0,1)$ distribution. Hence the price of the inflation-linked leg of the LPI swap at time 0 (note that when $M \geq 3$, it is only an approximation) is:

$$P_{0T^*}^N \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) \prod_{i=1}^{M} \left(\bar{F}_i - Call\left(\bar{F}_i, 1+C, \bar{\sigma}_i^2\right) + Put\left(\bar{F}_i, 1+F, \bar{\sigma}_i^2\right)\right) \mathrm{d}w.$$

It follows that we can value LPI swaps with just a single numerical integration.

IV. NUMERICAL EXAMPLES

We now examine some numerical examples. There are different forms that the volatility functions σ_{ktT}^N and σ_{jtT}^R can take, but here we will consider the extended Vasicek form in which we assume

$$\sigma_{ktT}^N = \frac{\sigma_k^N}{\alpha_k^N}\left(1 - \mathrm{e}^{-\alpha_k^N (T-t)}\right), \quad \sigma_{ktT}^R = \frac{\sigma_k^R}{\alpha_k^R}\left(1 - \mathrm{e}^{-\alpha_k^R (T-t)}\right), \tag{15}$$

where, for each k, σ_k^N, σ_k^R, α_k^N, and α_k^R are positive constants.

We will use the model parameters estimated for GBP in Li (2007). In order to simplify parameter estimation, we assume that real zero coupon bond prices are driven by a single Brownian motion so that $K_R = 1$ in equation (3). In addition, we assume that the volatility of the spot CPI is constant, i.e. $\sigma_t^X = \sigma^X$. We assume that there are two Brownian motions driving nominal zero coupon bond prices so that $K_N = 2$. This assumption adds nothing to the complexity of the calibration since the associated parameters can be (and were) obtained by calibrating to the market prices of GBP vanilla interest-rate swaptions (see Li 2007). The estimated values of the parameters are:

$$\begin{cases} \sigma_1^N = 0.00649825, & \alpha_1^N = 0.06494565, & \sigma_2^N = 0.0063321172, \\ \alpha_2^N = 0.00001557535, & \sigma_1^R = 0.006093904, & \alpha_1^R = 0.032193009, \\ \sigma^X = 0.0104000, & \rho_{12}^{NN} = -0.46296278, & \rho_1^{RX} = 0.03781752, \\ \rho_{11}^{NR} = \rho_{21}^{NR} = 0.518100, & \rho_1^{NX} = \rho_2^{NX} = 0.018398113 & \end{cases}$$

We will use these parameters to give some numerical examples and comparisons for inflation swaps with different swap tenors and payment times.

Example 1: The effect of the convexity adjustment on the fixed rate for zero coupon inflation swaps. Figure 1 shows the fixed rate K on zero coupon inflation swaps, with a payment delay of 5 years, for swaps of different tenors from 5 years to 25 years. The interest-rate (both nominal and real) yield curves were the GBP market implied rates as of June 2007 (see Appendix C for the set of market data). The volatility and correlation parameters were as above. The fixed rate on the swaps when we evaluate the convexity adjustment, using Proposition 1, is always lower than the fixed rate we would obtain on the swaps if we naively assumed that no convexity adjustment was necessary. Furthermore, the difference increases with increasing swap tenor. At 25 years, i.e. when $T_M = 25$ and $T_N = 30$,

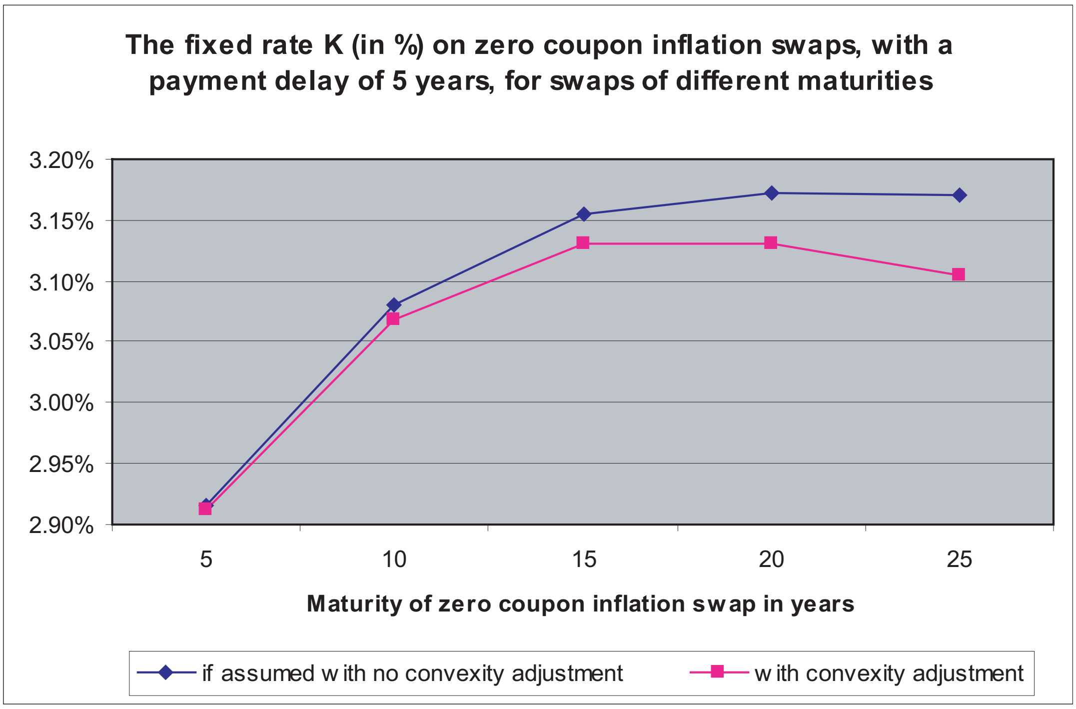

FIG. 1:

the difference is more than 0.065% which is, from a trader's perspective, significant as the bid-offer spread in the market, for zero coupon inflation swaps, is approximately 0.03%, or sometimes even less.

Some examples of period-on-period inflation swaps are provided in Li (2007) so here, in Examples 2 and 3, we will give some examples of the prices of LPI swaps, again using the volatility and correlation parameters above. For the purposes of these illustrations, we assumed, for both the examples below, that the interest-rate (both nominal and real) yield curves were initially flat and that nominal interest rates to all maturities were 0.05 and real interest rates to all maturities were 0.025, i.e. we assumed $P^N_{0T} = \exp(-0.05T)$ and $P^R_{0T} = \exp(-0.025T)$. We used Monte Carlo simulation with 130 million runs (65 million runs plus 65 million antithetic runs) in order to test and benchmark the accuracy of our application of the Ryten methodology.

Example 2: LPI swaps with floors and caps at $(0\%, 3\%)$, $(0\%, 5\%)$, $(1\%, 4\%)$. Here we consider three different combinations of floors and caps (which are commonly traded in the market), namely, $(0\%, 3\%)$, $(0\%, 5\%)$, and $(1\%, 4\%)$. For all three different combinations, we consider LPI swaps where each period is equal to one year, and the number of periods varies from one period, through 2, 5, 10, 15, 20, 25 to 30 periods and hence the maturities of the LPI swaps varied from one year to 30 years. We see from Figure 2 that the fixed rates obtained from the quasi-analytical methodology of Ryten (labelled QA) are very close to those obtained from Monte Carlo (labelled MC) simulation for shorter maturities (as explained above, the Ryten methodology is, in fact, essentially exact for $M \leq 2$). However, the differences do increase for LPI swaps with more periods.

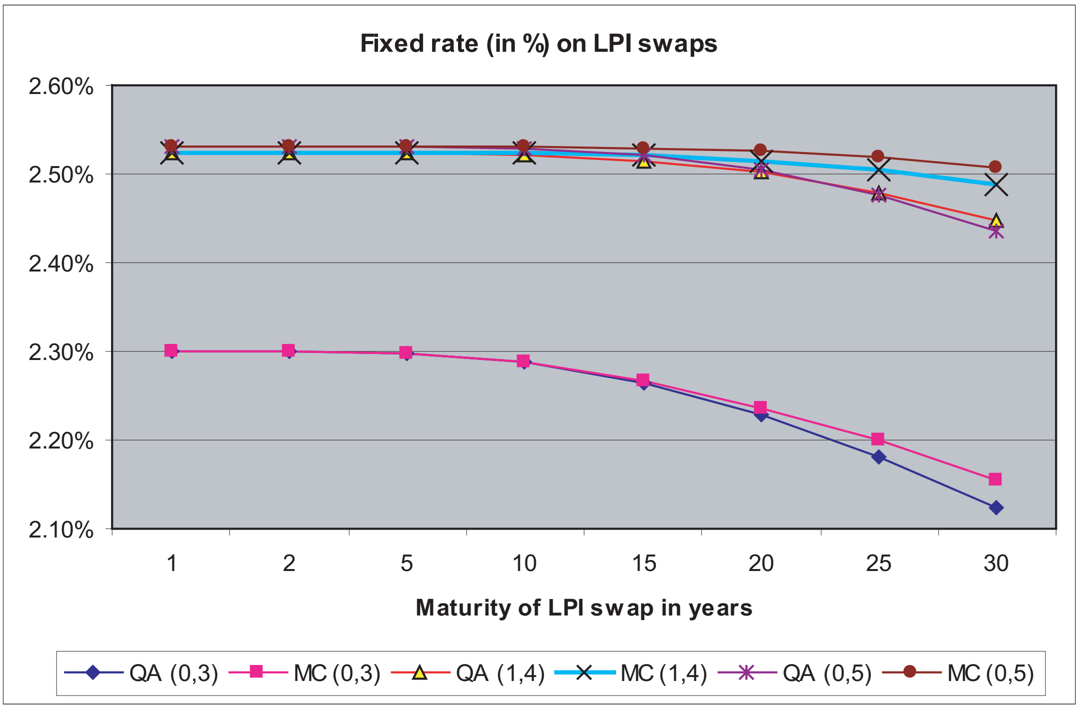

FIG. 2:

Example 3: LPI swaps with maturities of 10 years and 25 years . Here we consider eleven different combinations of floors and caps as indicated in Table 1. We consider LPI swaps whose maturities were 10 years and 25 years. Again, each period is equal to one year. We know that the Ryten methodology is essentially exact when $M \leq 2$. However, we see for the LPI swaps with 10 years maturity and 25 years maturity the level of approximation involved when $M \geq 3$. As a rough guide, the bid-offer spread in the market for LPI swaps is approximately 0.06% (expressed as the fixed rate on the swap). For the LPI swaps with 10 years maturity, the maximum difference (Table 1, 8th column) between the fixed rates implied by the Monte Carlo results (6th column) and by the Ryten methodology (7th column), is less than 0.0019%, which implies very accurate pricing as it is less than one thirtieth the typical bid-offer spread. For the LPI swaps with 25 years maturity, the accuracy does deteriorate somewhat. The maximum difference in the fixed rates is approximately 0.053%, which is close to the bid-offer spread.

Having given some examples of the valuation of LPI swaps, we can make one further comment about the accuracy of the quasi-analytical methodology. In tables 1 and 2, we observe that the accuracy deteriorates when the cap level is high and the floor level is low. This might initially seem surprising since in the limiting case that $C = \infty$ and $F = -\infty$ the LPI swaps become the same as standard zero coupon swaps. However, the reason for the deterioration in accuracy is that the quasi-analytical methodology approximates the correlation structure. Although (in the notation of Section III) it is true that $\mathbb{E}_0^{T^*}[\hat{X}_i] = \mathbb{E}_0^{T^*}[X_i]$ for all i, and it is also true that $\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M} X_i\right] = \mathbb{E}_0^{T^*}[X_{T_M}/X_0] = P_{0T_M}^R = P_{0T^*}^R$, the price of a standard zero coupon swap, the approximation of the correlation structure means that $\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M} \hat{X}_i\right]$ does *not* equal $\mathbb{E}_0^{T^*}\left[\prod_{i=1}^{M} X_i\right]$, except in the special cases for

10 year, 10 period LPI swap

Cap	floor	stan. error	Price Monte Carlo	Price Ryten (QA)	imp. rate MC %	imp. rate QA %	diff rates %
0.03	0	7.08E-06	0.760519	0.760461	2.28825	2.28746	0.00079
0.03	0.02	7.33E-06	0.777059	0.777044	2.50856	2.50836	0.00020
0.032	0.01	7.15E-06	0.767780	0.767724	2.38549	2.38475	0.00075
0.035	0.005	7.15E-06	0.770922	0.770840	2.42731	2.42622	0.00110
0.04	0.01	7.21E-06	0.778157	0.778063	2.52303	2.52179	0.00123
0.045	0.0175	7.33E-06	0.789247	0.789174	2.66821	2.66727	0.00094
0.0475	0.0025	7.21E-06	0.778593	0.778464	2.52878	2.52708	0.00170
0.05	0	7.21E-06	0.778669	0.778535	2.52978	2.52801	0.00177
0.05	0.005	7.21E-06	0.779410	0.779282	2.53953	2.53785	0.00169
0.06	0	7.21E-06	0.779061	0.778922	2.53493	2.53311	0.00183
0.12	-0.08	7.21E-06	0.778796	0.778654	2.53145	2.52957	0.00188

Table 1

25 year, 25 period LPI swap

cap	floor	stan. error	Price Monte Carlo	Price Ryten (QA)	imp. rate MC %	imp. rate QA %	diff rates %
0.03	0	1.62E-05	0.493509	0.491246	2.19897	2.18018	0.01879
0.03	0.02	1.87E-05	0.530458	0.529970	2.49455	2.49077	0.00378
0.032	0.01	1.69E-05	0.509992	0.508136	2.33336	2.31844	0.01492
0.035	0.005	1.66E-05	0.514297	0.511382	2.36778	2.34451	0.02327
0.04	0.01	1.71E-05	0.531668	0.528436	2.50389	2.47889	0.02500
0.045	0.0175	1.82E-05	0.557735	0.555077	2.70033	2.68071	0.01962
0.0475	0.0025	1.66E-05	0.533227	0.528129	2.51590	2.47651	0.03939
0.05	0	1.65E-05	0.533657	0.528121	2.51920	2.47645	0.04275
0.05	0.005	1.67E-05	0.536505	0.531414	2.54103	2.50193	0.03910
0.06	0	1.65E-05	0.536619	0.530530	2.54190	2.49511	0.04680
0.12	-0.08	1.63E-05	0.535254	0.528384	2.53145	2.47849	0.05297

Table 2

which $M \leq 2$.

For the sake of brevity, we only considered the Ryten methodology for the case of conditioning on one common factor. Ryten (2007) also considers the case of conditioning on two common factors (which means that evaluating the price of a LPI swap requires a double numerical integration) and shows, in his model set-up which is different to ours, that (unsurprisingly) this gives a significant improvement in accuracy. We would certainly conjecture that using two common factors would also significantly improve the accuracy of the prices of the LPI swaps which we reported in Tables 1 and 2. However, we leave confirmation of this conjecture for future research.

V. CONCLUSION

In recent years there has been a substantial increase in the demand for more exotic inflation derivative products. Working within a multi-factor version of the model of Hughston (1998) and Jarrow and Yildirim (2003), we have provided the economic rationale for, and the valuation formulae for, zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments. We have also valued LPI swaps, with the aid of the quasi-analytic methodology of Ryten (2007).

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References.

[1] Black F. (1976) "The pricing of commodity contracts" *Journal of Financial Economics* **3** p167-179.

[2] Crosby J. (2007) "Valuing inflation futures contracts" *Risk magazine* **March** p88-90.

[3] Dodgson M. and D. Kainth (2006) "Inflation-linked derivatives" Risk training course, 8th September 2006. Available at www.quarchome.org (Royal Bank of Scotland Quantitative Research Centre).

[4] Hughston L. P. (1998) "Inflation derivatives" Working paper, Merrill Lynch and King's College London (with added note 2004), available at www.mth.kcl.ac.uk/research/finmath.

[5] Jackel P. (2004) "Splitting the core" Working paper available at www.jaeckel.org/SplittingTheCore.pdf.

[6] Jarrow R. and Y. Yildirim (2003) "Pricing treasury inflation protected securities and related derivatives using an HJM Model" *Journal of Financial and Quantitative Analysis* **38** p409-430.

[7] Kerkhof J. (2005) "Inflation derivatives explained: Market, products, and pricing" Lehman Brothers publication.

[8] Li H. (2007) "Convexity adjustments in inflation-linked derivatives using a multi-factor version of the Jarrow and Yildirim model" M.Sc dissertation, Department of Mathematics, Imperial College London. Available online at: http://www.john-crosby.co.uk

[9] Mercurio F. (2005) “Pricing inflation-indexed derivatives” *Quantitative Finance* **5** p289-302.

[10] Ryten M. (2007) “Practical modelling for limited price index and related inflation products” Presentation given at the ICBI Global Derivatives conference in Paris on 22nd May 2007.

APPENDIX A: PROOF OF PROPOSITIONS 1 & 2

The stochastic discounting term $\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right)$ is log-normally distributed and can be written in the form

$$\begin{aligned}\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right) &= P_{tT_N}^N \exp\left(-\int_t^{T_N} \frac{1}{2}\sum_{k=1}^{K_N}\sum_{j=1}^{K_N} \rho_{kj}^{NN}\sigma_{ksT_N}^N\sigma_{jsT_N}^N \mathrm{d}s\right) \\ &\quad \times \exp\left(\int_t^{T_N}\sum_{k=1}^{K_N}\sigma_{ksT_N}^N \mathrm{d}z_{ks}^N\right).\end{aligned}$$

If we define the forward CPI at time t to time T by F_{tT}^X, then by no-arbitrage arguments, we have $F_{tT}^X = X_t(P_{tT}^R/P_{tT}^N)$, where F_{tT}^X is log-normally distributed (see, for example, Crosby 2007). Since

$$F_{T_M T_M}^X = X_{T_M}\frac{P_{T_M T_M}^R}{P_{T_M T_M}^N} = X_{T_M},$$

we find

$$\mathbb{E}_t\left[\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right) X_{T_M}\right] = \mathbb{E}_t\left[\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right) F_{T_M T_M}^X\right].$$

This expectation can be computed by noting that it is the expectation of a product of two log-normally distributed random variables, each of which has deterministic mean and variance terms. Li (2007) provides full details.

The proof of Proposition 2 is very similar to that for Proposition 1 except that now we will compute an expectation involving three log-normally distributed random variables.

APPENDIX B: INFLATION SWAPS WITH DELAYED PAYMENTS

In this appendix, we will value two types of inflation swap, namely, zero coupon inflation swaps with delayed payment and period-on-period inflation swaps with delayed payments. The key point about these types of inflation swap is that they have the same payoff as the corresponding inflation swap with no delayed payments but the payoff is paid at a later time. When the delay in payment is very small (for example, a few weeks), we would, intuitively, expect the difference between the values of the corresponding swaps with no delayed payments and with delayed payments to be small. Conversely, the difference in values can be substantial when the delay in payments is, for example, a few years. As we noted in Section I, inflation swaps with delayed payments of five years or more are quite commonly traded in the market.

1. Zero coupon inflation swaps with delayed payment

As mentioned above, it is now relatively common to trade zero coupon inflation swaps where the payment is delayed for some time, perhaps several years or more, compared to the payoff of a standard zero coupon inflation swap. Unlike with a standard (i.e. with no delayed payment) zero coupon inflation swap, the valuation of zero coupon inflation swaps with delayed payment will involve a convexity adjustment which is model-dependent. We can explicitly compute it within our model setup by using Proposition 1.

Suppose that today, at time 0, we enter into a zero coupon inflation swap with delayed payment. We denote the maturity of the swap by T_M and the payment time by $T_N \geq T_M$. We wish to value the swap, at time t, where $0 \leq t \leq T_M \leq T_N$. The payoff of the zero coupon inflation swap with delayed payment is still $N\left(X_{T_M}/X_0 - 1\right) - N\left((1+K)^{T_M} - 1\right)$, where K is the fixed rate on the swap and N is the notional amount, but this is paid at time $T_N \geq T_M$. The value, at time t, of the zero coupon inflation swap with delayed payment is:

$$\begin{aligned}
&\mathbb{E}_t\left[\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right)\left(N\left(\frac{X_{T_M}}{X_0} - 1\right) - N\left((1+K)^{T_M} - 1\right)\right)\right] \\
&\quad = \mathbb{E}_t\left[N\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right)\left(\frac{X_{T_M}}{X_0} - (1+K)^{T_M}\right)\right] \\
&\quad = \frac{N}{X_0}\,\mathbb{E}_t\left[X_{T_M}\exp\left(-\int_t^{T_N} r_s^N \mathrm{d}s\right)\right] - NP^N_{tT_N}(1+K)^{T_M} \\
&\quad = \frac{N}{X_0}\,X_t P^R_{tT_M}\frac{P^N_{tT_N}}{P^N_{tT_M}}\exp\left(\int_t^{T_M} C_s(T_M,T_N)\mathrm{d}s\right) - NP^N_{tT_N}(1+K)^{T_M}. \qquad \text{(B1)}
\end{aligned}$$

Note that in obtaining the last line we have used Proposition 1. Compared to the value of a standard zero coupon inflation swap with no delayed payment, we see that there is an extra term $(P^N_{tT_N}/P^N_{tT_M})\,\mathrm{e}^{\int_t^{T_M} C_s(T_M,T_N)\mathrm{d}s}$ in the inflation-linked leg.

2. Period-on-period inflation swaps with delayed payments

Our aim now is to value, at any time t, a period-on-period inflation swap with delayed payments. Proposition 2 will be the key to this.

Suppose that today, at time 0, we enter into a period-on-period inflation swap with delayed payments. The swap is defined via a set of fixed dates $T_0 < T_1 < T_2 < \cdots < T_{M-1} < T_M$, where $T_0 = 0$. These dates are usually approximately one year apart but they need not be. As with a standard interest-rate swap, a period-on-period inflation swap is made up of a series of swaplets. The key issue is that the value of the payoff of each swaplet is the same as the payoff of the corresponding swaplet of a period-on-period inflation swap with no delayed payments but now the payment is made at time T_{N_i} which is some time greater than or equal to T_i. The payoff of the ith swaplet, for $i = 1, 2, \ldots, M$, at time T_{N_i}, is $N\tau_i^I\left(X_{T_i}/X_{T_{i-1}} - 1\right) - N\tau_i^F K$, where K is the fixed rate on the swap, N is the notional amount, τ_i^I is the day-count adjusted time from T_{i-1} to T_i for the floating (inflation-linked) leg, and τ_i^F is the day-count adjusted time from T_{i-1} to T_i for the fixed leg.

The value at time t of the swaplet with delayed payment ($T_{N_i} \geq T_i$) is

$$\begin{aligned}
&\mathbb{E}_t\left[\exp\left(-\int_t^{T_{N_i}} r_s^N \mathrm{d}s\right)\left(N\tau_i^I\left(\frac{X_{T_i}}{X_{T_{i-1}}} - 1\right) - N\tau_i^F K\right)\right] \\
&\quad = N\tau_i^I\,\mathbb{E}_t\left[\exp\left(-\int_t^{T_{N_i}} r_s^N \mathrm{d}s\right)\frac{X_{T_i}}{X_{T_{i-1}}}\right] - NP^N_{tT_{N_i}}(\tau_i^I + \tau_i^F K). \qquad \text{(B2)}
\end{aligned}$$

To value the floating (inflation-linked) side, we have to consider separately two different cases depending upon whether $t \geq T_{i-1}$ or $t < T_{i-1}$.

In the first case for which $T_{i-1} \leq t \leq T_i$, the value of $X_{T_{i-1}}$ is known at time t. Therefore, we can take $X_{T_{i-1}}$ outside of the expectation in (B2) and obtain, from Proposition 1,

$$\mathbb{E}_t \left[\exp\left(-\int_t^{T_{N_i}} r_s^N \mathrm{d}s \right) X_{T_i} \right] = X_t P_{tT_i}^R \frac{P_{tT_{N_i}}^N}{P_{tT_i}^N} \exp\left(\int_t^{T_i} C_s(T_i, T_{N_i}) \mathrm{d}s \right). \tag{B3}$$

Substitution of this expression in the right side of (B2) then yields

$$N\tau_i^I \frac{X_t}{X_{T_{i-1}}} P_{tT_i}^R \frac{P_{tT_{N_i}}^N}{P_{tT_i}^N} \exp\left(\int_t^{T_i} C_s(T_i, T_{N_i}) \mathrm{d}s \right) - NP_{tT_{N_i}}^N (\tau_i^I + \tau_i^F K).$$

In the second case for which $0 \leq t < T_{i-1}$ we use the result of Proposition 2 to obtain

$$N\tau_i^I P_{tT_{i-1}}^N \frac{P_{tT_i}^R}{P_{tT_{i-1}}^R} \frac{P_{tT_{N_i}}^N}{P_{tT_i}^N} \exp\left(\int_{T_{i-1}}^{T_i} C_s(T_i, T_{N_i}) \mathrm{d}s + \int_t^{T_{i-1}} \left[A_s(T_{i-1}, T_i) + B_s(T_{i-1}, T_i, T_{N_i}) \right] \mathrm{d}s \right)$$
$$- NP_{tT_{N_i}}^N (\tau_i^I + \tau_i^F K).$$

Therefore, we can value a period-on-period inflation swap with delayed payments by summing up the value of all the swaplets, bearing in mind the two distinct expressions arising from the case $T_{i-1} \leq t \leq T_i$ and the case $0 \leq t < T_{i-1}$.

Note that when $T_i = T_{N_i}$, $B_s(T_{i-1}, T_i, T_{N_i})$ and $C_s(T_i, T_{N_i})$ in (9) and (6) vanish. Hence, one can confirm, after some algebra, that the results we have just given, in the case of extended Vasicek bond volatilities (see equation (15)), reduce to the same as those given in Mercurio (2005) for the value of a period-on-period inflation swap with no delayed payments.

APPENDIX C: MARKET DATA FOR EXAMPLE 1

Tenor	Nominal Discount Factors	Real Discount Factors
5	0.747665196	0.863178385
10	0.574072261	0.777518375
15	0.450566319	0.717981039
20	0.361027914	0.674313663
25	0.301528182	0.657905735
30	0.242028449	0.614217677