Caps and Floors

John Crosby

Glasgow University

My website is: http://www.john-crosby.co.uk
If you spot any typos or errors, please email me.
My email address is on my website

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Introduction and motivation

- Caps and floors are amongst the simplest and most common types of interest-rate derivatives.
- They are very widely used by corporations for managing interest-rate risk.
- Individuals also buy caps (perhaps without knowing it) when they buy capped mortgages.
- In this lecture:
- 1./ We will recap different definitions and meanings of interest-rates (spot interest-rates, LIBOR rates, forward interest-rates) and their relationships to zero-coupon bonds.
- 2./ We will analyse caps and floors and derive pricing relationships.
- 3./ We will also examine the relationship between forward contracts and futures contracts focussing on the difference between a 3 month FRA (forward-rate agreement) and a 3 month LIBOR (Eurodollar) futures contract.

Assumptions

- We will make some key assumptions (which are essentially the same as those used to derive the Black-Scholes option pricing formula for stock options) throughout this lecture:
- The key assumption we make is that we assume the absence of arbitrage.
- We also assume that markets are frictionless eg continuous trading (in zero-coupon bonds) is possible, there are no transactions costs (i.e. no bid-offer spreads, it is possible to borrow and lend at the same rate), etc.

Recap on interest-rate definitions

- Calendar time is denoted by t.
- We let the price, at time t, of a zero-coupon bond which matures at time T be $P(t, T)$.
- Note three important (but I hope obvious) things:
- Firstly, $P(t, T)$ is known when the valuation date is on or after t.
- Secondly, by contrast, $P(t, T)$ for any t greater than the valuation date is a stochastic quantity.
- Thirdly, $P(T, T) = 1$, for any T.
- We denote the (continuously-compounded) instantaneous short interest-rate, at time t, by $r(t)$. Note that $r(t)$ will, throughout this lecture, be stochastic.

Introduction to LIBOR

- In the interest-rate markets, one of the main interest-rates quoted is the LIBOR rate.
- LIBOR is quoted as a simple (i.e. without compounding) interest-rate and is the rate that a bank will lend money to another credit-worthy institution (although we draw no distinction between the bank lending and the bank accepting deposits) from time T_1 to time T_2. (We will not, in this lecture, consider credit-related issues).
- We let τ be the time in years between T_1 and T_2 but adjusted for the day-count fraction. So roughly, $\tau \approx T_2 - T_1$.
- In practice, τ is often approximately 0.25 or 0.50 (corresponding to 3 months or 6 months) but LIBOR is quoted for periods from overnight to one year.

Introduction to LIBOR 2

- Suppose today is time T_1. We are given the price $P(T_1, T_2)$ (ie at time T_1 - this is the meaning of the first argument of P) of a zero-coupon bond which matures at time T_2. We are given a LIBOR rate $L(T_1, T_1, T_2)$, at time T_1, from time T_1 to time T_2 (the meaning of the two arguments equal to T_1 will be clear shortly). What is the relationship between $L(T_1, T_1, T_2)$ and $P(T_1, T_2)$?

- It is:

$$P(T_1, T_2) = \frac{1}{1 + \tau L(T_1, T_1, T_2)} \qquad \text{or } L(T_1, T_1, T_2) = \frac{1}{\tau}\Big(\frac{1}{P(T_1, T_2)} - 1\Big), \tag{1}$$

where, again, τ is the time in years between T_1 and T_2 (but adjusted for the day-count fraction: $\tau \approx T_2 - T_1$).

Introduction to spot, short interest-rates

- Today is still time T_1. We are still given the price $P(T_1, T_2)$ of a zero-coupon bond which matures at time T_2. We are given a continuously-compounded spot interest-rate $R(T_1, T_2)$ from time T_1 to time T_2. What is the relationship between $R(T_1, T_2)$ and $P(T_1, T_2)$?

- It is:

$$P(T_1, T_2) = \exp(-R(T_1, T_2)(T_2 - T_1)) \qquad \text{or } R(T_1, T_2) = \frac{-1}{(T_2 - T_1)} \log P(T_1, T_2). \tag{2}$$

- What is the relationship between the (continuously-compounded) instantaneous short interest-rate $r(t)$, at time t, for $t \in [T_1, T_2]$ and $P(T_1, T_2)$. It is:

$$P(T_1, T_2) = \mathbb{E}^{\mathbb{Q}}_{T_1}[\exp(-\int_{T_1}^{T_2} r(u)du)]. \tag{3}$$

  Here, $\mathbb{E}^{\mathbb{Q}}_{T_1}[\bullet]$ denotes expectations, at time T_1, under some risk-neutral measure $\mathbb{Q}$.

Introduction to spot, short interest-rates 2

- Again:

$$P(T_1, T_2) = \mathbb{E}^{\mathbb{Q}}_{T_1}[\exp(-\int_{T_1}^{T_2} r(u)du)]. \tag{4}$$

- Note that: $P(T_1, T_2) \neq \exp(-\int_{T_1}^{T_2} r(u)du)$. ($P(T_1, T_2)$ would be equal to $\exp(-\int_{T_1}^{T_2} r(u)du)$ in the special case that $r(t)$ is deterministic - but we won't be assuming that).

- Note that equation (4) has a simple interpretation: Standard theory tells us that prices are expected (under $\mathbb{Q}$) discounted payoffs. The zero-coupon bond pays one at time T_2, so:

$$P(T_1, T_2) = \mathbb{E}^{\mathbb{Q}}_{T_1}[\exp(-\int_{T_1}^{T_2} r(u)du) \ \mathbf{1} \ ], \tag{5}$$

  in agreement with equation (4).

Introduction to spot, short interest-rates 3

- The spot interest-rate $R(T_1, T_2)$ from time T_1 to time T_2 can be interpreted as an average interest-rate over the time period $[T_1, T_2]$ and also as the yield on the zero-coupon bond which matures at time T_2.

- The instantaneous short interest-rate $r(t)$ should be interpreted as a stochastic process. We know what $r(t)$ is at time t. For $u > t$, $r(u)$ is only known as calendar time advances to time u. Then the instantaneous short interest-rate $r(u)$ is known and can be interpreted as the interest-rate on a bank account over the infinitesimally short time period from u to $u + du$.

Forward LIBOR rate

- We denote the LIBOR rate observed, at time t, which sets at time T_1 and is paid at time T_2 by $L(t, T_1, T_2)$, with $t < T_1 < T_2$. But what does it mean "observed at time t" since the LIBOR rate itself will only be known at time T_1 (with $T_1 > t$) (and the actual interest payment will be paid at time T_2 (with $T_2 > T_1 > t$))?

Forward LIBOR rate replication 1

- How can I replicate, at time t, receiving a guaranteed payment of $L(T_1, T_1, T_2)$ at time T_2 where I will only know the value of $L(T_1, T_1, T_2)$ at time T_1, with $T_1 > t$? Consider the following strategy:

- At time t: I buy $1/\tau$ zero-coupon bonds maturing at time T_1 for a price of $P(t, T_1)/\tau$ and I simultaneously sell $1/\tau$ zero-coupon bonds maturing at time T_2 for a price of $P(t, T_2)/\tau$.

- At time T_1, my T_1 maturity bonds mature and I place the principal amount of $1/\tau$ on deposit at the (now known) rate of $L(T_1, T_1, T_2)$. I will get this back with interest at time T_2 and receive (principal + interest) $(1 + \tau L(T_1, T_1, T_2))/\tau$. I will have to pay $1/\tau$ at time T_2 to repay the T_2 maturity bonds which mature. Therefore at time T_2, I receive a net amount of:

$$((1 + \tau L(T_1, T_1, T_2))/\tau) - (1/\tau) = L(T_1, T_1, T_2).$$

Forward LIBOR rate replication 2

- In short, I receive a guaranteed (in the assumed absence of credit-risk) payment of $L(T_1, T_1, T_2)$ at time T_2 from my trades. How much did it cost me to enter into the trades? At time t, I paid $P(t, T_1)/\tau$ to buy $1/\tau$ zero-coupon bonds maturing at time T_1 and I received $P(t, T_2)/\tau$ for selling $1/\tau$ zero-coupon bonds maturing at time T_2. So the net cost, at time t, was:

$$(P(t, T_1)/\tau) - (P(t, T_2)/\tau) = (P(t, T_1) - P(t, T_2))./\tau$$

- Therefore, in the assumed absence of arbitrage, the value, at time t, of receiving a payment of $L(T_1, T_1, T_2)$ at time T_2, is: $(P(t, T_1) - P(t, T_2))/\tau$. More mathematically:

$$\mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_2} r(s)ds)L(T_1, T_1, T_2)] = (P(t, T_1) - P(t, T_2))/\tau. \tag{6}$$

Here, $\mathbb{E}_t^Q[\bullet]$ denotes expectation, at time t, under the risk-neutral equivalent martingale measure $\mathbb{Q}$. We recall that $r(s)$ is the (continuously-compounded) instantaneous short interest-rate, at time s.

Forward LIBOR rate replication 3

- The previous slide gives the answer to our question on slide "Forward LIBOR rate". Given the prices, at time t, of zero-coupon bonds maturing at time T_1 and at time T_2, and a simple trading strategy, we can perfectly replicate, a payment of $L(T_1, T_1, T_2)$ at time T_2. We do not have to wait until time T_1 when the LIBOR rate becomes known.

- Therefore, we define $L(t, T_1, T_2)$, for $t < T_1$, via:

$$L(t, T_1, T_2) = \frac{(P(t, T_1) - P(t, T_2))/\tau}{P(t, T_2)} = \frac{1}{\tau}(\frac{P(t, T_1)}{P(t, T_2)} - 1).$$

- This is the same as:

$$\frac{1}{1 + \tau L(t, T_1, T_2)} = \frac{P(t, T_2)}{P(t, T_1)}.$$

FRA - forward-rate agreement

- The last two slides define an instrument called a FRA. A FRA is a forward contract on forward LIBOR i.e. on $L(t, T_1, T_2)$. It has a payoff at time T_2 equal to:

  $$L(T_1, T_1, T_2) - K,$$

  where K is chosen so that the contract costs nothing to enter into at time t.

- Therefore, K must be equal to: $K = L(t, T_1, T_2)$.

- FRA's are very actively traded in the markets. One will hear them referred to as, for examples, the 3 X 6 FRA. It means T_1 is (approximately) 3 months from t and T_2 is (approximately) 6 months from t.

Caps

- A caplet is an option which has a payoff at time T_2 equal to:

$$\max(L(T_1, T_1, T_2) - K, 0),$$

  where K is the strike.

- Note the sequence of times implicit in a caplet: The LIBOR rate sets (i.e. becomes known) at time T_1. This means the payoff of the caplet is known at time T_1 also. The payoff is actually made at some later time T_2 (i.e. $T_2 > T_1$) which is also the payment time of the LIBOR rate. We are interested in the price of the caplet at some time t. To be interesting, we assume that $t < T_1$.

- Before discussing the valuation of caplets. We briefly discuss some practicalities.

Practicalities of caps

- Notional amounts of fifty million pounds (or equivalent in dollars or euros) are fairly standard in the caps market.

- Caps are a linear combination of caplets with dates defined via $T_0 < T_1 < T_2 < \ldots < T_N$. Therefore, we focus on caplets since given a pricing formula for caplets, we can price caps trivially by adding together the price of each individual caplet. When the sequence of dates is such that the payoff of the first caplet is known (or will be known) on the date the cap is entered into, the convention is to disregard the first caplet. So a semi-annual two year cap, entered into on 6th Feb 2009, with final maturity 7th Feb 2011, only has three underlying caplets (with payoff dates 8th February 2010, 9th August 2010 and 7th February 2011).

- Floors are the "put" version of caps and can similarly be decomposed into linear combinations of floorlets. A floorlet has a payoff at time T_2 equal to:

$$\max(K - L(T_1, T_1, T_2), 0).$$

Practicalities of caps 2

- We will discuss the pricing of caps shortly.

- As with options on other asset classes (such as equities), prices are quoted in the market in the form of Black implied volatilities (the meaning of which, we discuss shortly). As with other asset classes, these implied volatilities display considerable variation with maturity and strike. In other words, we see skews and smiles in implied volatilities as well as significant time-dependence.

- Caps for maturities 1y, 18m, 2y, 3y, 4y, 5y, 10y, 15y, 20y, 25y and 30y trade at a variety of strikes from close to zero per cent to, perhaps, around 5 per cent above the prevailing forward LIBOR rate. The most liquid are close to at-the-money but there is reasonable liquidity even in caps which are significantly out-of-the-money.

Caplet 1

- A caplet is an option which has a payoff at time T_2 equal to:

$$\max(L(T_1, T_1, T_2) - K, 0).$$

- Note the sequence of times implicit in a caplet: The LIBOR rate sets (i.e. becomes known) at time T_1. This means the payoff of the caplet is known at time T_1 also. The payoff is actually made at some later time T_2 (i.e. $T_2 > T_1$) which is also the payment time of the LIBOR rate.

- Since

$$\frac{1}{(1 + \tau L(T_1, T_1, T_2))} = P(T_1, T_2),$$

  we can write:

$$L(T_1, T_1, T_2) = \frac{1}{\tau}(\frac{1}{P(T_1, T_2)} - 1).$$

  Again, τ is the day-count adjusted time in years between T_1 and T_2.

Caplet 2

- Therefore, the caplet payoff at time T_2 is:

$$\max(L(T_1, T_1, T_2) - K, 0) = \max(\frac{1}{\tau}(\frac{1}{P(T_1, T_2)} - 1) - K, 0).$$

- Now lets pause. The caplet payoff is known at time T_1 but made at time T_2. The price of a zero-coupon bond, at time T_1, maturing at time T_2 is $P(T_1, T_2)$.

- Therefore, the caplet payoff $\max(L(T_1, T_1, T_2) - K, 0)$ at time T_2 is completely equivalent to a payoff of $P(T_1, T_2) \max(L(T_1, T_1, T_2) - K, 0)$ at time T_1. This must be true because, at time T_1, one could simply place the known amount of $P(T_1, T_2) \max(L(T_1, T_1, T_2) - K, 0)$ on deposit or in zero-coupon bonds maturing at time T_2. At time T_2, one would have $\max(L(T_1, T_1, T_2) - K, 0)$. But

$$\begin{aligned} P(T_1, T_2) \max(L(T_1, T_1, T_2) - K, 0) &= P(T_1, T_2) \max(\frac{1}{\tau}(\frac{1}{P(T_1, T_2)} - 1) - K, 0) \\ &= \frac{1}{\tau} \max(1 - (1 + K\tau) P(T_1, T_2), 0). \end{aligned}$$

Caplet 3

- So a caplet with a payoff at time T_2 equal to $\max(L(T_1, T_1, T_2) - K, 0)$ is equivalent to an option with a payoff at time T_1 equal to:
$$\frac{1}{\tau}\max(1 - (1 + K\tau)P(T_1, T_2), 0).$$
But this is just the same as $(1 + K\tau)/\tau$ options with strike $K^\star \equiv 1/(1 + K\tau)$ paying
$$\max(K^\star - P(T_1, T_2), 0) \text{ at time } T_1\ .$$
But $\max(K^\star - P(T_1, T_2), 0)$ paid at time T_1 is the same as the payoff, at time T_1, of a put option on the price, at time T_1, of a zero-coupon bond maturing at time T_2, where the strike of the put option is $K^\star$.

- So a caplet is a put option on a zero-coupon bond. So to price a caplet, we need to be able to price an option on a zero-coupon bond. We will do this shortly.

Caplet 4

- How do we price an option on a zero-coupon bond?

- We will now answer this question. It is somewhat similar to the derivation of the Black-Scholes formula for an option on a stock but with a key difference: Bond prices and interest-rates are stochastic.

- We assume that the dynamics of the price $P(t,T)$ of a zero coupon bond, at time t, maturing at time T_i (here, i is 1 or 2) are:

$$\frac{dP(t,T_i)}{P(t,T_i)} = \mu_i(t)dt + \sigma_P(t,T_i)dz_i^{\mathbb{P}}(t). \tag{7}$$

- It is intuitive that, in a risk-neutral world, the drift term $\mu_i(t)$ is equal to $r(t)$ (after all, a zero-coupon bond is a non-dividend-paying asset) but we will not yet make that identification. We will assume that the SDE above specifies the dynamics in the real-world objective probability measure $\mathbb{P}$ (hence the superscript $\mathbb{P}$ in $dz_i^{\mathbb{P}}(t)$ in the SDE) and go through a hedging argument to derive the equivalent of the Black-Scholes PDE.

Caplet 5

- We consider an option which pays, at time T_1, an amount $C(T_1)$ (for example, $C(T_1) \equiv \max(P(T_1,T_2)-K,0)$ or $C(T_1) \equiv \max(K-P(T_1,T_2),0)$).

- We denote the price, at time t, of the option by $C(t)$. It is intuitive that the price $C(t)$ actually depends on both $P(t,T_1)$ and $P(t,T_2)$. That is to say:

$$C(t) \equiv C(t, P(t,T_1), P(t,T_2)).$$

Using the multi-dimensional form of Ito's lemma and writing $P(t,T_1) \equiv P_1$ and $P(t,T_2) \equiv P_2$:

$$\begin{aligned} dC(t) &= \frac{\partial C(t)}{\partial t}dt + \frac{\partial C(t)}{\partial P_1}(\mu_1 dt + \sigma_P(t,T_1)dz_1^{\mathbb{P}}(t))P_1 + \frac{\partial C(t)}{\partial P_2}(\mu_2 dt + \sigma_P(t,T_2)dz_2^{\mathbb{P}}(t))P_2 \\ &+ \Big(\frac{1}{2}\frac{\partial^2 C(t)}{\partial P_1^2}\sigma_P^2(t,T_1)P_1^2 + \frac{1}{2}\frac{\partial^2 C(t)}{\partial P_2^2}\sigma_P^2(t,T_2)P_2^2 \\ &+ \rho_{12}\frac{\partial^2 C(t)}{\partial P_1 \partial P_2}\sigma_P(t,T_1)\sigma_P(t,T_2)P_1P_2\Big)dt. \end{aligned}$$

Here ρ_{12} is the (assumed deterministic) instantaneous correlation between $dz_1^{\mathbb{P}}(t)$ and $dz_2^{\mathbb{P}}(t)$.

Caplet 6

- We set up a self-financing portfolio consisting, at time t, of long one option, short $\frac{\partial C(t)}{\partial P_1}$ bonds maturing at time T_1 and short $\frac{\partial C(t)}{\partial P_2}$ bonds maturing at time T_2. We do not hold any position in cash at any time. The portfolio always has zero net aggregate investment.

- The value of the portfolio at time t is

$$0 = C(t) - \frac{\partial C(t)}{\partial P_1} P_1 - \frac{\partial C(t)}{\partial P_2} P_2.$$

  The left-hand-side of the last equation is simply the statement of zero net aggregate investment, for all t.

- We stress that the short positions in bonds maturing at time T_1 and at time T_2 are dynamic positions which are changing through time (analogously to the dynamic delta-hedge used in deriving the Black-Scholes pde for stock options).

Caplet 7

- The change in the value of the portfolio, during the period t to $t + dt$ is:

$$\begin{aligned} dC(t) - \frac{\partial C(t)}{\partial P_1} dP_1 - \frac{\partial C(t)}{\partial P_2} dP_2 &= \frac{\partial C(t)}{\partial t} dt \\ &+ \Big(\frac{1}{2} \frac{\partial^2 C(t)}{\partial P_1^2} \sigma_P^2(t, T_1) P_1^2 + \frac{1}{2} \frac{\partial^2 C(t)}{\partial P_2^2} \sigma_P^2(t, T_2) P_2^2 \\ &+ \rho_{12} \frac{\partial^2 C(t)}{\partial P_1 \partial P_2} \sigma_P(t, T_1) \sigma_P(t, T_2) P_1 P_2 \Big) dt, \end{aligned}$$

where we used slide "Caplet 5". Note that the terms with $dz_1^{\mathbb{P}}(t)$ and $dz_2^{\mathbb{P}}(t)$ have cancelled so this is an instantaneously risk-free portfolio. The portfolio is always worth zero - because it always has zero net aggregate investment. Therefore, in the absence of arbitrage, it must have a realised return of exactly zero, for all t.

- Therefore:

$$dC(t) - \frac{\partial C(t)}{\partial P_1} dP_1 - \frac{\partial C(t)}{\partial P_2} dP_2 = 0.$$

Caplet 8

- Or:

$$
\begin{aligned}
0 &= \frac{\partial C(t)}{\partial t} \\
&+ \frac{1}{2}\frac{\partial^2 C(t)}{\partial P_1^2}\sigma_P^2(t,T_1)P_1^2 + \frac{1}{2}\frac{\partial^2 C(t)}{\partial P_2^2}\sigma_P^2(t,T_2)P_2^2 \\
&+ \rho_{12}\frac{\partial^2 C(t)}{\partial P_1 \partial P_2}\sigma_P(t,T_1)\sigma_P(t,T_2)P_1P_2.
\end{aligned}
$$

- This is a pde - essentially, the equivalent of the Black-Scholes pde.

- We would like to reduce it to a pde in one state variable. How can we do this?

Caplet 9

- We guess (and show later that it is a good guess) that the price of the option $C(t) \equiv C(t, P(t,T_1), P(t,T_2))$ is actually a function of the forward bond price defined by:

$$F(t) \equiv F(t,T_2) \equiv \frac{P(t,T_1)}{P(t,T_2)} \equiv \frac{P_1}{P_2}.$$

Specifically, we write:

$$C(t) \equiv C(t, P(t,T_1), P(t,T_2)) \equiv P(t,T_2)G(t,F(t)) \equiv P_2G(t).$$

Then:

$$\begin{aligned}
\frac{\partial C(t)}{\partial P_1} &= P_2 \frac{\partial G(t)}{\partial F}\frac{\partial G(t)}{\partial P_1} = P_2 \frac{\partial G(t)}{\partial F}\frac{1}{P_2} = \frac{\partial G(t)}{\partial F}, \\
\frac{\partial C(t)}{\partial P_2} &= G(t) + P_2 \frac{\partial G(t)}{\partial F}\frac{\partial F}{\partial P_2} = G(t) + P_2 \frac{\partial G(t)}{\partial F}\left(\frac{-P_1}{P_2^2}\right) = G(t) - G(t)\frac{\partial G(t)}{\partial F}, \\
\frac{\partial^2 C(t)}{\partial P_1^2} &= \frac{\partial^2 G(t)}{\partial F^2}\frac{1}{P_2}, \qquad \frac{\partial^2 C(t)}{\partial P_2^2} = F^2 \frac{\partial^2 G(t)}{\partial F^2}\frac{1}{P_2}, \\
\frac{\partial^2 C(t)}{\partial P_1 P_2} &= -F \frac{\partial^2 G(t)}{\partial F^2}\frac{1}{P_2}.
\end{aligned}$$

Caplet 10

- Let us check our equation for zero net aggregate investment:

$$0 = C(t) - \frac{\partial C(t)}{\partial P_1} P_1 - \frac{\partial C(t)}{\partial P_2} P_2.$$

The right-hand-side becomes:

$$\begin{aligned} C(t) - \frac{\partial C(t)}{\partial P_1} P_1 - \frac{\partial C(t)}{\partial P_2} P_2 &= P_2 G(t) - \frac{\partial G(t)}{\partial F} P_1 - \left(G(t) + P_2 \frac{\partial G(t)}{\partial F} \left(\frac{-P_1}{P_2^2} \right) \right) P_2 \\ &= 0. \end{aligned} \tag{8}$$

So our zero net aggregate investment condition is automatically satisfied, for all t.

- The pde becomes:

$$\begin{aligned} 0 &= P_2 \frac{\partial G(t)}{\partial t} + \frac{1}{2} P_2 \frac{\partial^2 G(t)}{\partial F^2} \sigma_P^2(t, T_1) F^2 \\ &+ \frac{1}{2} P_2 \frac{\partial^2 G(t)}{\partial F^2} \sigma_P^2(t, T_2) F^2 - P_2 \rho_{12} \frac{\partial^2 G(t)}{\partial F^2} \sigma_P(t, T_1) \sigma_P(t, T_2) F^2, \end{aligned}$$

which is the same as:

$$0 = \frac{\partial G(t)}{\partial t} + \frac{1}{2} \Sigma^2(t) \frac{\partial^2 G(t)}{\partial F^2} F^2, \tag{9}$$

where:

$$\Sigma^2(t) \equiv \Big(\sigma_P^2(t, T_1) + \sigma_P^2(t, T_2) - \rho_{12} \sigma_P(t, T_1) \sigma_P(t, T_2) \Big).$$

This is just the Black-Scholes pde with zero interest-rates, zero dividend yield and volatility $\Sigma(t)$.

Forward measure

- We know that we can write the price of the option, at time t, in the form:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_1} r(s)ds)C(T_1)].$$

- However, from the pde in equation (9), we can also evaluate $G(t)$ by writing it in the form:

$$G(t) = \mathbb{E}_t^{\mathbb{Q}^\star}[G(T_1)] = \mathbb{E}_t^{\mathbb{Q}^\star}[G(T_1, \frac{P(T_1,T_1)}{P(T_1,T_2)})], \tag{10}$$

  where, $\mathbb{Q}^\star$ defines a new probability measure (called the forward measure) in which we form expectations "as if" interest-rates and dividend yields in the Black-Scholes pde are zero (the former explains why there is no term of the form $\exp(-\int_t^{T_1} r(s)ds)$ in equation (10)).

- However, $C(t) = P(t,T_2)G(t)$ and $C(T_1) = P(T_1,T_2)G(T_1)$. Therefore:

$$C(t) \;=\; P(t,T_2)\mathbb{E}_t^{\mathbb{Q}^\star}[G(T_1)] = P(t,T_2)\mathbb{E}_t^{\mathbb{Q}^\star}[\frac{C(T_1)}{P(T_1,T_2)}]. \tag{11}$$

Forward measure 2

- Suppose, we consider a payoff, at time T_1, equal to $P(T_1, T_2)$. Since $P(T_1, T_2)$ paid at time T_1 is the same as 1 paid at time T_2, we know that, trivially, $C(t) = P(t, T_2)$. From our pricing formulae, we get:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_1} r(s)ds)P(T_1, T_2)] \text{ and } C(t) = P(t, T_2)\mathbb{E}_t^{\mathbb{Q}^\star}[\frac{P(T_1, T_2))}{P(T_1, T_2)}] = P(t, T_2).$$

So:

$$P(t, T_2) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_1} r(s)ds)P(T_1, T_2)] = P(t, T_2),$$

which is simply confirmation of what we already know. It also confirms that, under the risk-neutral equivalent martingale measure $\mathbb{Q}$, the drift rate on a zero-coupon bond, at time t, is $r(t)$.

Forward measure 3

- More pertinently, we consider a payoff, at time T_1, equal to 1. We know that, trivially, $C(t) = P(t, T_1)$. From our pricing formulae, we get:

$$C(t) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_1} r(s)ds)\, 1] \text{ and } C(t) = P(t, T_2)\mathbb{E}_t^{\mathbb{Q}^\star}[\frac{1}{P(T_1, T_2)}].$$

So:

$$P(t, T_1) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_1} r(s)ds)\, 1] = P(t, T_2)\mathbb{E}_t^{\mathbb{Q}^\star}[\frac{1}{P(T_1, T_2)}].$$

Since $P(T_1, T_1) = 1$, let us write the last equation:

$$\frac{P(t, T_1)}{P(t, T_2)} = \mathbb{E}_t^{\mathbb{Q}^\star}[\frac{P(T_1, T_1)}{P(T_1, T_2)}].$$

- This means that in the measure $\mathbb{Q}^\star$, the quantity $P(t, T_1)/P(t, T_2)$ is a martingale - the value of the quantity at time t is the expectation of its value at time T_1. This is the first time you will have encountered the idea of a change of measure in this way. Our approach has been long-winded but it can be made simpler when you have studied Girsanov's theorem.

Forward measure 4

- Again:

$$\frac{P(t,T_1)}{P(t,T_2)} = \mathbb{E}_t^{\mathbb{Q}^\star}[\frac{P(T_1,T_1)}{P(T_1,T_2)}].$$

- Since, $1+\tau L(t,T_1,T_2) = P(t,T_1)/P(t,T_2)$ and $1+\tau L(T_1,T_1,T_2) = P(T_1,T_1)/P(T_1,T_2)$. We can also write:

$$L(t,T_1,T_2) = \mathbb{E}_t^{\mathbb{Q}^\star}[L(T_1,T_1,T_2)]. \tag{12}$$

- This means that in the measure $\mathbb{Q}^\star$, the forward LIBOR rate is also a martingale.

- Recall, that in equation (6):

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_2} r(s)ds)L(T_1,T_1,T_2)] &= (P(t,T_1)-P(t,T_2))/\tau, \text{ hence rearranging} \\ L(t,T_1,T_2) &= \frac{\mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_2} r(s)ds)L(T_1,T_1,T_2)]}{P(t,T_2)}. \qquad (13) \end{aligned}$$

Forward measure 5

- Again:

$$
\begin{aligned}
L(t, T_1, T_2) &= \mathbb{E}_t^{\mathbb{Q}^\star}[L(T_1, T_1, T_2)]. \\
L(t, T_1, T_2) &= \frac{\mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^{T_2} r(s)ds) L(T_1, T_1, T_2)]}{P(t, T_2)}.
\end{aligned}
$$

- Note how the forward LIBOR rate $L(t, T_1, T_2)$ is a martingale under the measure $\mathbb{Q}^\star$ but NOT under the risk-neutral equivalent martingale measure $\mathbb{Q}$.

- Note further that the second expression for $L(t, T_1, T_2)$ is much more complicated since it involves an expectation under $\mathbb{Q}$ of the two product of two stochastic quantities $\exp(-\int_t^{T_2} r(s)ds)$ and $L(T_1, T_1, T_2)$ and, furthermore, these quantities are certainly NOT mathematically independent.

Summary so far

- Let us summarize and review our analysis so far:

- It is possible to perfectly hedge an option on a zero-coupon bond by dynamic trading in zero coupon bonds of two different maturities.

- Under the risk-neutral equivalent martingale measure $\mathbb{Q}$, zero-coupon bonds have a drift, at time t, equal to $r(t)$.

- Under the forward measure $\mathbb{Q}^{\star}$, the forward bond price and the forward LIBOR rate are martingales. They are NOT martingales under the risk-neutral equivalent martingale measure $\mathbb{Q}$.

Caplet pricing formulae

- We will now derive two specific caplet pricing formulae under two different assumptions for the dynamics of bond prices or forward LIBOR rates.
- Note that these formulae are not consistent with one another. They are consistent with the assumptions made to derive them but these assumptions are different for the two cases.
- The first specific caplet pricing formula comes from making the assumption that zero-coupon bond prices have a deterministic (lognormal) volatility. In other words, in equation (7), the terms $\sigma_P(t, T_i)$, for i equals 1 or 2 are deterministic and zero-coupon bond prices are lognormally distributed.

First caplet pricing formula - lognormal bond prices - 1

- We wanted to price an option on a zero-coupon bond paying $\max(K^\star - P(T_1, T_2), 0)$ at time T_1 but so as to be able to price both call and put options, we consider a payoff $\max(\eta(K^\star - P(T_1, T_2)), 0)$ at time T_1, where η is 1 or -1. In other words,

$$C(T_1) = \max(\eta(K^\star - P(T_1, T_2)), 0).$$

But $C(t) \equiv P(t, T_2)G(t)$. Therefore, from (9):

$$\begin{aligned} G(T_1) &= C(T_1)/P(T_1, T_2) = \max(\eta(K^\star - P(T_1, T_2)), 0)/P(T_1, T_2) \\ &= K^\star \max\big(\eta(\frac{1}{P(T_1, T_2)} - \frac{1}{K^\star}), 0\big) \\ &= K^\star \max\big(\eta(\frac{P(T_1, T_1)}{P(T_1, T_2)} - \frac{1}{K^\star}), 0\big), \quad \text{since } P(T_1, T_1) \equiv 1 \\ &= K^\star \max(\eta(F(T_1, T_2) - \frac{1}{K^\star}), 0). \end{aligned}$$

- Therefore, the payoff in terms of $G(T_1)$ is just like $K^\star$ call (if $\eta = 1$) options on the forward price struck at $1/K^\star$. We can write down the price in terms of $G(t)$ because it is the same as the Black-Scholes formula with zero interest-rates, zero dividend yield and volatility $\Sigma(t)$.

First caplet pricing formula - lognormal bond prices - 2

- This uses the fact that, because the bond volatilities $\sigma_P(t, T_i)$ are deterministic, so is $\Sigma(t)$.
- The option price in terms of $G(t)$ is:

$$
\begin{aligned}
G(t) &= \eta K^\star \big(F(t, T_2) N(\eta d_1) - (1/K^\star) N(\eta d_2)\big) \\
&= \eta K^\star \big(\frac{P(t, T_1)}{P(t, T_2)} N(\eta d_1) - (1/K^\star) N(\eta d_2)\big).
\end{aligned}
$$

- Note that (conveniently) this equation does NOT involve $r(t)$.

First caplet pricing formula - lognormal bond prices - 3

- The option price in terms of $C(t) \equiv P(t,T_2)G(t)$ is:

$$\begin{aligned} C(t) &= P(t,T_2)\eta K^\star\Big(\frac{P(t,T_1)}{P(t,T_2)}N(\eta d_1) - (1/K^\star)N(\eta d_2)\Big) \\ C(t) &= K^\star\eta\Big(P(t,T_1)N(\eta d_1) - (1/K^\star)P(t,T_2)N(\eta d_2)\Big) \\ C(t) &= \eta(K^\star P(t,T_1)N(\eta d_1) - P(t,T_2)N(\eta d_2)), \end{aligned}$$

where

$$d_1 = \frac{\log((P(t,T_1)/P(t,T_2))/(1/K^\star)) + \frac{1}{2}\hat{\Sigma}^2(T_1-t)}{\hat{\Sigma}(T_1-t)^{1/2}}, \; d_2 = d_1 - \hat{\Sigma}(T_1-t)^{1/2}.$$

and $\hat{\Sigma}$ is defined by:

$$\hat{\Sigma}^2(T_1-t) = \int_t^{T_1} \Big(\sigma_P^2(s,T_1) + \sigma_P^2(s,T_2) - \rho_{12}\sigma_P(s,T_1)\sigma_P(s,T_2)\Big)ds.$$

First caplet pricing formula - lognormal bond prices - 4

- From slide "Caplet 3", a caplet is equivalent to $(1 + K\tau)/\tau$ options with strike $K^\star \equiv 1/(1 + K\tau)$ paying $\max(K^\star - P(T_1, T_2), 0)$ at time T_1. So from above, the caplet price, at time t, is:

$$\begin{aligned}
&((1 + K\tau)/\tau)\eta(K^\star P(t, T_1)N(\eta d_1) - P(t, T_2)N(\eta d_2)) \\
=\ &((1 + K\tau)/\tau)\eta((1/(1 + K\tau))P(t, T_1)N(\eta d_1) - P(t, T_2)N(\eta d_2)) \\
=\ &(1/\tau)\eta(P(t, T_1)N(\eta d_1) - (1 + K\tau)P(t, T_2)N(\eta d_2)) \\
=\ &(1/\tau)\eta P(t, T_2)((1 + \tau L(t, T_1, T_2))N(\eta d_1) - (1 + \tau K)N(\eta d_2)). \qquad (14)
\end{aligned}$$

First caplet pricing formula - lognormal bond prices - 5

- Now all we need is a specific form for the bond volatilities $\sigma_P(t, T_i)$:
- There is more than one possibility for this. We will limit ourselves to mentioning only one.
- In the extended Vasicek model (also known as a one factor Hull and White model or a one factor Gaussian Heath, Jarrow and Morton model), we have:

$$\sigma_P(t, T_i) = \frac{\overline{\sigma}_r}{\alpha}(1 - \exp(-\alpha(T_i - t))),$$

  where $\overline{\sigma}_r$ and α are positive constants. It is a one-factor model, so $\rho_{12} = 1$. Pricing a caplet is now a trivial exercise in high-school algebra in computing $\hat{\Sigma}^2$.

Discussion on first caplet pricing formula

- The extended Vasicek (Hull and White) model has a number of attractive features (which you will learn about in detail on another day). It is a no-arbitrage model which is automatically consistent with any initial term-structure of interest-rates and it can be used to, very efficiently, price a very wide range of exotic (i.e. non-standard) interest-rate derivatives. However, in terms of pricing caps, the formula above is, in practice, mostly used in reverse: One uses the formula above in calibrating the extended Vasicek model i.e. to determine the values of the model parameters ($\overline{\sigma}_r$ and α) which give the best fit to the market prices of caps (in practice, vanilla (standard European) swaptions are also often used).

- The market-standard model is the Black (1976) model.

- The Black (1976) model assumes that the forward LIBOR rate underlying the cap is lognormally distributed with volatility σ_L.

Second caplet pricing formula - Black (lognormal LIBOR rates) - 1

- We have already seen that under the forward measure $\mathbb{Q}^\star$, the forward LIBOR rate is a martingale.

- We assume it follows a SDE under $\mathbb{Q}^\star$ as follows:

$$\frac{dL(t, T_1, T_2)}{L(t, T_1, T_2)} = \sigma_L dz^{\mathbb{Q}^\star}(t). \tag{15}$$

- We know the caplet payoff $\max(L(T_1, T_1, T_2) - K, 0)$ at time T_2 is completely equivalent to a payoff of $P(T_1, T_2)\max(L(T_1, T_1, T_2) - K, 0)$ at time T_1. Now, we use equation (11):

$$\begin{aligned} C(t) &= P(t, T_2)\mathbb{E}_t^{\mathbb{Q}^\star}\left[\frac{P(T_1, T_2)\max(L(T_1, T_1, T_2) - K, 0)}{P(T_1, T_2)}\right] \\ &= P(t, T_2)\mathbb{E}_t^{\mathbb{Q}^\star}\big[\max(L(T_1, T_1, T_2) - K, 0)\big]. \end{aligned} \tag{16}$$

Second caplet pricing formula - Black (lognormal LIBOR rates) - 2

- Equations (15) and (16) and comparison with the standard Black-Scholes formula enable us to write down the pricing formula immediately. The caplet price, at time t, is:

$$C(t) = P(t,T_2)(L(t,T_1,T_2)N(d_1) - KN(d_2)),$$

where, here,

$$d_1 = \frac{\log(L(t,T_1,T_2)/K) + \frac{1}{2}\sigma_L^2(T_1 - t)}{\sigma_L(T_1 - t)^{1/2}} \quad \text{and } d_2 = d_1 - \sigma_L(T_1 - t)^{1/2}.$$

- Note that $P(t,T_2)$ appears as the "discounting term". In the formulae for d_1 and d_2, it is T_1 which appears. This is intuitive - the "optionality" finishes at time T_1 (since the payoff will then be known with certainty) although the actual payoff is not paid until time T_2.

Discussion on second caplet pricing formula i.e. Black (1976)

- As already remarked, the market-standard model is the Black (1976) model. It is, in fact, more than a model - it is the market price quotation system. In other words, cap prices are quoted in the market as Black (1976) implied volatilities for a stated maturity and strike. Because these implied volatilities, in practice, vary with strike (which is contrary to what is implied by the assumptions of the model), there is a sense in which the Black (1976) model represents "the wrong volatility number to put into wrong formula to give the right price". Nonetheless, it is the market-standard model and the LIBOR market model (also known as the BGM model (after Brace, Gatarek and Musiela)) is based upon it.

3 month LIBOR futures contracts

- We have already seen that a FRA is a forward contract on a LIBOR rate. We will now discuss futures contracts on LIBOR rates. We will refer to them as 3 month LIBOR futures contracts (sometimes they are called 3 month Eurodollar futures contracts but the term Eurodollar is a slight misnomer as such contracts occur in all the major currencies (USD, EUR, STG, AUD, CAD, JPY, etc)). In practice, all these contracts are on 3 month LIBOR (or an equivalent 3 month rate) but there is nothing special about the 3 month term and our analysis does not depend on it.

- We mention a few practicalities first:

- These 3 month LIBOR futures contracts are extremely actively traded. The amounts traded are huge and the bid-offer spreads are tiny (often equivalent to 0.005 on a futures price of, say, 96.55 eg. the difference between an implied LIBOR interest-rate of, say, 3.450 % and 3.455 %).

- In USD, 3 month LIBOR futures contracts trade with a futures maturity of upto ten years (and in other currencies, for upto five years or more).

3 month LIBOR futures contracts 2

- How do 3 month LIBOR futures contracts work?
- The delivery (or settlement) price is 100 minus a 3 month LIBOR rate expressed as a percentage.
- We denote the futures contract maturity by T_1. At time T_1, the LIBOR rate $L(T_1, T_1, T_2)$ underlying the futures contract is observed. Then $100(1 - L(T_1, T_1, T_2))$ is the delivery (or settlement) price of the futures contract. For 3 month LIBOR futures contracts, $\tau \equiv T_2 - T_1$ is approximately 0.25 years although we will not assume that. Our aim is to derive the theoretical value of this futures contract, at time t, with $t < T_1$.
- Before we can do that, we need to fully understand how futures contracts work.

3 month LIBOR futures contracts 3

- Like a forward contract, it costs nothing up-front to enter into a futures contract. However, whereas with a forward contract, a single exchange of cashflows takes place at maturity, with futures contracts, there is a continuous (in practice, daily) mark-to-market procedure.

- An example helps: Suppose we buy a futures contract on 8th February when its market futures price is 97.21. Suppose its end-of-day settlement futures price (i.e. the closing futures price) on the exchange on 8th February is 97.33. An amount of $97.33 - 97.21 = 0.12$ (i.e. our gain) is paid into our account on that day by the exchange. Suppose we keep the position and its end-of-day settlement futures price on the exchange on 9th February is 97.18. We have to pay an amount of $97.33 - 97.18 = 0.15$ to the exchange on that day. Suppose we keep the position and its end-of-day settlement futures price on the exchange on 10th February is 97.07. We have to pay an amount of $97.18 - 97.07 = 0.11$ to the exchange on that day. (To clarify even more, this means we will have paid a net amount of $-0.12 + 0.15 + 0.11 = 0.14 = 97.21 - 97.07$ to the exchange by that time).

- This mark-to-market procedure continues until the futures contract matures at time T_1.

3 month LIBOR futures contracts 4

- Our aim is to derive the theoretical value of this futures contract, at time t, with $t < T_1$. In other words, what value should be placed at time t to enter into a futures contract whose delivery price, at time T_1, is $100(1 - L(T_1, T_1, T_2))$.

- Note two further key points:

- 1./ We use the word value rather than price to emphasise that is costs nothing up-front to enter into a futures contract.

- 2./ Obviously, it is the market that sets the market futures price. We are aiming to derive the theoretical value as it will show key points about futures contracts and their relationship to forward contracts.

- We let the theoretical value of the futures contract, at time t, be $G(t, T_1, T_2)$.

- The specification of the delivery price implies:

$$G(T_1, T_1, T_2) = 100(1 - L(T_1, T_1, T_2)) = 100(1 - \frac{1}{\tau}\big(\frac{1}{P(T_1, T_2)} - 1\big)). \tag{17}$$

- As with pricing FRA's and caplets, we will go through a replication or hedging argument. In fact, the derivation is very similar to that for caplets.

- It is intuitive that $G(t, T_1, T_2)$ actually depends on both $P(t, T_1)$ and $P(t, T_2)$. That is to say:

$$G(t) \equiv G(t, T_1, T_2) \equiv G(t, P(t, T_1), P(t, T_2)).$$

Using the multi-dimensional form of Ito's lemma and writing $P(t, T_1) \equiv P_1$ and $P(t, T_2) \equiv P_2$:

$$\begin{aligned} dG(t) &= \frac{\partial G(t)}{\partial t}dt + \frac{\partial G(t)}{\partial P_1}(\mu_1 dt + \sigma_P(t, T_1)dz_1^{\mathbb{P}}(t))P_1 + \frac{\partial G(t)}{\partial P_2}(\mu_2 dt + \sigma_P(t, T_2)dz_2^{\mathbb{P}}(t))P_2 \\ &+ \Big(\frac{1}{2}\frac{\partial^2 G(t)}{\partial P_1^2}\sigma_P^2(t, T_1)P_1^2 + \frac{1}{2}\frac{\partial^2 G(t)}{\partial P_2^2}\sigma_P^2(t, T_2)P_2^2 \\ &+ \rho_{12}\frac{\partial^2 G(t)}{\partial P_1 \partial P_2}\sigma_P(t, T_1)\sigma_P(t, T_2)P_1 P_2\Big)dt. \end{aligned}$$

As before, ρ_{12} is the (assumed deterministic) instantaneous correlation between $dz_1^{\mathbb{P}}(t)$ and $dz_2^{\mathbb{P}}(t)$.

- We set up a self-financing portfolio consisting, at time t, of long one futures contract, short $\frac{\partial G(t)}{\partial P_1}$ bonds maturing at time T_1 and short $\frac{\partial G(t)}{\partial P_2}$ bonds maturing at time T_2. We do not hold any position in cash at any time (which means that, for example, any gains from the futures contract are invested in bonds). The portfolio always has zero net aggregate investment.

- The value of the portfolio at time t is:

$$0 = 0 - \frac{\partial G(t)}{\partial P_1} P_1 - \frac{\partial G(t)}{\partial P_2} P_2. \tag{18}$$

  The left-hand-side of the last equation is simply the statement of zero net aggregate investment, for all t. The 0 on the right-hand-side is the statement that there is no up-front cost to buying (or selling) a futures contract.

- It is clear that, in order to satisfy equation (18), $\frac{\partial G(t)}{\partial P_1}$ and $\frac{\partial G(t)}{\partial P_2}$ must be of different sign.

- The change in the value of the portfolio during the period t to $t + dt$ is:

$$
\begin{aligned}
dG(t) - \frac{\partial G(t)}{\partial P_1} dP_1 - \frac{\partial G(t)}{\partial P_2} dP_2 &= \frac{\partial G(t)}{\partial t} dt \\
&+ \Big(\frac{1}{2} \frac{\partial^2 G(t)}{\partial P_1^2} \sigma_P^2(t, T_1) P_1^2 + \frac{1}{2} \frac{\partial^2 G(t)}{\partial P_2^2} \sigma_P^2(t, T_2) P_2^2 \\
&+ \rho_{12} \frac{\partial^2 G(t)}{\partial P_1 \partial P_2} \sigma_P(t, T_1) \sigma_P(t, T_2) P_1 P_2 \Big) dt,
\end{aligned}
$$

where we used slide "3 month LIBOR futures contracts 6". Note that the terms with $dz_1^{\mathbb{P}}(t)$ and $dz_2^{\mathbb{P}}(t)$ have cancelled so this is an instantaneously risk-free portfolio. The portfolio is always worth zero - because it always has zero net aggregate investment. Therefore, in the absence of arbitrage, it must have a realised return of exactly zero, for all t.

- Therefore:

$$
dG(t) - \frac{\partial G(t)}{\partial P_1} dP_1 - \frac{\partial G(t)}{\partial P_2} dP_2 = 0.
$$

- Therefore:

$$
\begin{aligned}
0 &= \frac{\partial G(t)}{\partial t} \\
&+ \frac{1}{2}\frac{\partial^2 G(t)}{\partial P_1^2}\sigma_P^2(t,T_1)P_1^2 + \frac{1}{2}\frac{\partial^2 G(t)}{\partial P_2^2}\sigma_P^2(t,T_2)P_2^2 \\
&+ \rho_{12}\frac{\partial^2 G(t)}{\partial P_1 \partial P_2}\sigma_P(t,T_1)\sigma_P(t,T_2)P_1P_2.
\end{aligned}
$$

- Note that this is the same pde as we derived for caplets (although the zero net aggregate investment condition is different).

- We must solve it together with the conditions defined by equations (17) and (18).

- To do this, we make the assumption that zero-coupon bond prices have a deterministic (lognormal) volatility (eg. as in the extended Vasicek model).

- We guess a solution of the form:

$$G(t,T_1,T_2) = 100(1 - \frac{1}{\tau}\Big(\frac{P(t,T_1)}{P(t,T_2)}\exp(\Upsilon(t,T_1,T_2)) - 1\Big)),$$

where $\Upsilon(t) \equiv \Upsilon(t,T_1,T_2)$ is a purely deterministic quantity (to be found).

- Then we have:

$$\frac{\partial G(t)}{\partial t} = \frac{1}{\tau}\Big(\frac{\partial \Upsilon(t)}{\partial t}\frac{P(t,T_1)}{P(t,T_2)}\exp(\Upsilon(t,T_1,T_2))\Big), \qquad \frac{\partial^2 G(t)}{\partial P_1^2} = 0,$$

$$\frac{\partial^2 G(t)}{\partial P_2^2} = \frac{1}{\tau}\Big(\frac{2P(t,T_1)}{P(t,T_2)^3}\exp(\Upsilon(t,T_1,T_2))\Big), \quad \frac{\partial^2 G(t)}{\partial P_1 P_2} = \frac{1}{\tau}\Big(\frac{-1}{P(t,T_2)^2}\exp(\Upsilon(t,T_1,T_2))\Big).$$

- Substituting into the pde (and simplifying) we get:

$$0 = \frac{\partial \Upsilon(t)}{\partial t} + \sigma_P^2(t, T_2) - \rho_{12}\sigma_P(t, T_1)\sigma_P(t, T_2).$$

  Hence,

$$\Upsilon(t) = \int_t^{T_1} \Big(\sigma_P^2(s, T_2) - \rho_{12}\sigma_P(s, T_1)\sigma_P(s, T_2)\Big) ds.$$

- Once we substitute the specific forms for $\sigma_P(t, T_1)$ and $\sigma_P(t, T_2)$ (eg. from the extended Vasicek model on slide "First caplet pricing formula - lognormal bond prices - 5"), we can trivially compute a specific form for $\Upsilon(t)$.

- It is straightforward to verify that equations (17) and (18) are satisfied. So, our solution is:

$$G(t, T_1, T_2) = 100(1 - \frac{1}{\tau}\Big(\frac{P(t, T_1)}{P(t, T_2)} \exp(\Upsilon(t, T_1, T_2)) - 1\Big)).$$

- We make two important points.
- The first point shows that $G(t, T_1, T_2)$ is a martingale under the risk-neutral equivalent martingale measure $\mathbb{Q}$. To see this, define $H(t) \equiv H(t, T_1, T_2)$ via:

$$H(t, T_1, T_2) \equiv \frac{P(t, T_1)}{P(t, T_2)} \exp(\Upsilon(t, T_1, T_2)).$$

- Under $\mathbb{Q}$, the dynamics of zero coupon bond prices are:

$$\frac{dP(t, T_i)}{P(t, T_i)} = r(t)dt + \sigma_P(t, T_i)dz_i^{\mathbb{Q}}(t). \tag{19}$$

- Then, by Ito's lemma, (after a little algebra - I want you to prove this at the computer lab) the dynamics of $H(t)$ are:

$$\frac{dH(t)}{H(t)} = \sigma_P(t, T_1)dz_1^{\mathbb{Q}}(t) - \sigma_P(t, T_2)dz_2^{\mathbb{Q}}(t). \tag{20}$$

- Therefore, $H(t)$ is a martingale under $\mathbb{Q}$ and hence $G(t, T_1, T_2)$ is also martingale under the risk-neutral equivalent martingale measure $\mathbb{Q}$.

- We saw in slide "Summary so far", that the forward LIBOR rate $L(t, T_1, T_2)$ is a martingale under the measure $\mathbb{Q}^\star$ but NOT under the risk-neutral equivalent martingale measure $\mathbb{Q}$.

- By contrast, $G(t, T_1, T_2)$ (which is the value of a futures contract on forward LIBOR) is a martingale under the risk-neutral equivalent martingale measure $\mathbb{Q}$ (but NOT under the measure $\mathbb{Q}^\star$).

- This is a highly intuitive result.

- The second point is as follows:

- Note that if interest-rates were to be deterministic, then $\sigma_P(t, T_1)$ and $\sigma_P(t, T_2)$ would be zero and $\Upsilon(t)$ would be zero so that $G(t, T_1, T_2)$ would be equal to $100(1 - L(t, T_1, T_2))$ - in other words, if interest-rates were to be deterministic then the value of the futures contract would be essentially (ignoring constant factors) the same as a FRA rate (which is the fair rate on a forward contract on forward LIBOR).

- However, in general, we can write:

$$\Upsilon(t) = -\int_t^{T_1} \mathrm{Covar}\left(\frac{dP(s, T_2)}{P(s, T_2)}, \frac{dH(s)}{H(s)}\right) ds.$$

- We certainly expect $\Upsilon(t)$ to be positive because we expect bond prices of different maturities to be positively correlated.

- Therefore, we expect $G(t, T_1, T_2)$ to be less than $100(1 - L(t, T_1, T_2))$.

- Let us think about the intuition in being long a futures contract. If the market futures price goes up, then I will be paid the price change by the exchange which I can then place on deposit. If the market price goes down, then I will have to borrow in order to pay the price change to the exchange. But the interest-rate that I will get on my deposit or pay on my borrowing is negatively correlated with the market futures price (since the market futures price is essentially proportional to one minus an interest-rate).

- Therefore, on average, when market futures price goes up, I will receive a lower rate on deposit for my gains and, on average, when market futures price goes down, I will have to pay a higher interest-rate on my borrowings to fund my losses.

- Therefore, on average, I seem to lose whichever way interest-rates move.

- In an arbitrage-free market, I expect this to be reflected in the prices in the market.
- Therefore, I expect $G(t, T_1, T_2)$ to be less than $100(1 - L(t, T_1, T_2))$, which is what we have already seen.
- Note that this effect is purely a function of interest-rate volatility (and its consequences for correlations between changes in interest-rates and changes in market futures prices) - it has got nothing to do with, for example, the shape of the yield curve.
- This effect is known as the convexity adjustment or convexity bias.
- The effect is very significant, especially for long-dated futures contracts i.e. those for which T_1 is quite large (we have already noted 3 month LIBOR futures contracts trade with a futures maturity of upto ten years i.e. $T_1 - t$ can be approximately 10). In the computer lab, we will quantify this effect.

Futures contracts to build yield curves

- We have derived the theoretical value of a 3 month LIBOR futures contract. Of course, the market sets the market futures price. The theoretical value is most useful in solving the reverse problem: Given $\Upsilon(t, T_1, T_2)$ and given the market futures price, what does it imply for forward LIBOR rates and hence for discount factors. In other words, the market futures prices of 3 month LIBOR futures contracts (with different T_1) are used to build yield-curves.

- If I fail to take into account the effects of the convexity adjustment term (i.e. of $\Upsilon(t, T_1, T_2)$), then my yield-curve will be wrong. Then I will also misprice FRA's, swaps and other interest-rate derivatives.

Futures contracts to build yield curves 2

- In the early 1990's, a number of financial institutions are reputed to have lost a lot of money (and others to have made a lot of money (reputed to be $>$ 500 million US dollars)) by failing to recognise the necessity of incorporating the convexity adjustment term when valuing 3 month LIBOR futures contracts or when building yield-curves from market futures prices.

- The financial institutions that lost a lot of money were effectively (and erroneously) always assuming that $\Upsilon(t, T_1, T_2) = 0$.

Summary and General Conclusions

- We have priced FRA's, caplets and 3 month LIBOR futures contracts.
- As with pricing stock options, the key to pricing these instruments is to construct replicating portfolios or hedges and then the assumption of no-arbitrage gives us pricing formulae.
- Foward LIBOR rates are martingales under the forward measure $\mathbb{Q}^\star$ whereas futures contracts on forward LIBOR rates are martingales under the risk-neutral equivalent martingale measure $\mathbb{Q}$.
- The tool of changing measure from $\mathbb{Q}$ to $\mathbb{Q}^\star$ is a vital tool that you will come across in greater detail later in your course.