Pricing a class of exotic commodity options in a multi-factor jump-diffusion model

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Abstract

A recent paper, Crosby (2005), introduced a multi-factor jump-diffusion model which would allow futures (or forward) commodity prices to be modelled in a way which captured empirically observed features of the commodity and commodity options markets. However, the model focused on modelling a single individual underlying commodity. In this paper, we investigate an extension of this model which would allow the prices of multiple commodities to be modelled simultaneously in a simple but realistic fashion. We then price a class of simple exotic options whose payoff depends on the difference (or ratio) between the prices of two different commodities (for example, spread options), or between the prices of two different (ie with different tenors) futures contracts on the same underlying commodity, or between the prices of a single futures contract as observed at two different calendar times (for example, forward start or cliquet options). We show that it is possible, using a Fourier Transform based algorithm, to derive a single unifying form for the prices of all these aforementioned exotic options and some of their generalisations. Although we focus on pricing options within the model of Crosby (2005), most of our results would be applicable to other models where the relevant "extended" characteristic function is available in analytical form.

1. Introduction

Our aim, in this paper, is to price a class of simple European-style exotic commodity options within an extension of the Crosby (2005) model. One of the features of the commodities markets is that options which are considered "exotic" for other asset classes are very common in the commodities markets. Consider an option which pays the greater of zero and the difference between the prices of two commodities minus a fixed strike (which might in practice, be zero). These options are very actively traded. When the commodities are crude oil and a refined oil product (such as heating oil or jet fuel), an option on the price difference is called a crack spread option. These crack spread options are actively traded, not only in the OTC market but also, on NYMEX, the New York futures exchange. When one of the commodities is coal, spread options are called dark spread options and when one of the commodities is electricity, spread options are called spark spread options. Phraseology apart, all these options are options on the difference between the prices of two commodities. The prices in question might be the futures prices to some given tenors or the spot prices of two different commodities. In this paper, we will focus on the case when the prices in question are futures commodity prices because, we can easily include the case of spot prices as a special case of the former (ie as a futures contract which matures at the same time as the option maturity).

Another phraseology that is also used for spread options is that of "primary" commodity and "daughter" commodity. A "primary" commodity might be, for example, a very actively traded blend of crude oil (in practice, either Brent or WTI) and a "daughter" commodity would then be either a much less actively traded blend (eg Bonny Light from Nigeria or Dubai) of crude oil or a refined petroleum product such as heating oil, jet fuel or gasoline. The price movements of the "daughter" commodity would closely, but not perfectly, follow those of the "primary" commodity. In practice, many spread options involve a "primary" commodity and a "daughter" commodity although, clearly, spread options on two seemingly unrelated commodities, such as natural gas and a base metal, are possible.

It would be possible to approximate the prices of spread options by making ad-hoc assumptions such as assuming the price spread is normally or log-normally distributed. However, such assumptions are ad-hoc and are inconsistent with the assumptions typically made about the dynamics of the individual commodities. The disadvantages of these approaches are discussed in, for example, Dempster and Hong (2000) and Garman (1992). We will briefly mention one disadvantage of modelling price spreads ie (arithmetic) price differences as log-normal. Because, of the way that crude oil is refined, through fractional distillation, into a basket of refined products, one would expect the basket of refined products to be always worth more than the same quantity of crude oil and that the difference is the (positive) cost of refining. One might therefore be tempted to expect that the price of a particular refined petroleum product (for example, heating oil or aviation fuel) is always higher (when measured in the same units) than the price of crude oil. In fact, whilst a positive price differential is the more common situation, it is empirically observed (see, for example, Geman (2005)) that sometimes an imbalance of supply and demand in the international markets results in a negative price differential, albeit usually for just short periods of time. It is also observed that, over a period of time, the spot price of a benchmark grade of crude oil (such as Brent or WTI) can trade both more expensively and, at different times, more cheaply than a given, less actively traded grade of crude oil (such as Bonny Light or Dubai). Clearly, it would not be appropriate, therefore, to model (arithmetic) price differences as log-normal. So therefore, in this paper, we will look at pricing spread options without ad-hoc assumptions about the price spread and consistent with each of the two commodities following the dynamics of the model of Crosby (2005).

Quite often the fixed strike of the spread option is, in fact, zero and we will call these "zero strike" spread options. These are the type we will focus on, in this paper. The "zero strike" type of spread option (an option to exchange one asset for another) was first considered by Margrabe (1978) for the case of log-normally distributed asset prices. See also Rubinstein (1991a), (1991b) and Geman (2005) and the references therein. Duffie et al. (2000) consider the pricing of some simple types of exotic options for assets (bonds (both risk-free and defaultable), foreign exchange rates and equities) following affine jump-diffusion processes. Deng (1998) considers the pricing of spread options on spot commodity prices where the underlying spot commodity prices follow affine jump-diffusion processes. In addition, Dempster and Hong (2000) have considered spread options (including the more difficult case of "non-zero-strike") on options where the underlying assets can follow more general stochastic processes, including processes with stochastic volatility. Duffie et al. (2000), Deng (1998) and Dempster and Hong (2000) all use Fourier Transform methods.

There are other actively traded variants on spread options, including options on the price ratio (rather than the price difference). Another variant is that the underlying is actually a single physical commodity but the spread involves the price difference (or ratio) between two futures contracts on that same commodity but with two different tenors. These could be viewed as options on the slope of the futures commodity curve. A somewhat different variant again is that a single commodity futures contract is observed at two different calendar times. This gives rise to forward start and ratio forward start (single leg cliquet) options. Using Fourier Transform methods, we will derive a single unifying form for all these exotic options and some of their generalisations.

It is well-known (see Geman (2005) and Crosby (2005) and the references therein) that jumps are an important feature of the commodities and commodity options markets, being both more frequent and larger in magnitude than in, for example, the equity and foreign exchange markets.

In Crosby (2005), we introduced a multi-factor jump-diffusion model for commodities and commodity options. It is an arbitrage-free model consistent with any initial term structure of futures commodity prices. The model incorporates multiple jump processes into the dynamics of futures commodity prices. It also allows for a specific empirically observed feature, common in the commodities markets (especially for energy related commodities such as crude oil, natural gas and electricity), that when there are jumps in futures commodity prices, the short end of the futures commodity price curve jumps by a larger magnitude than the long end of the futures commodity price curve. This is a feature that did not seem to have appeared in the literature before. In fact, Deng (1998), and several other papers, such as Hilliard and Reis (1998) and Clewlow and Strickland (2000), include jumps in models for spot commodity prices. None of these models are consistent with any initial term structure of futures commodity prices but even if time-dependent drift terms were introduced to allow for this, they are only able to produce jumps which cause parallel shifts in the term structure of (log) futures commodity prices. We also allow for these latter types of jumps (see Assumption 2.2 in section 2) but, in addition, through an exponential dampening feature, we also allow for jumps (see Assumption 2.1 in section 2) which cause long-dated futures commodity prices to jump by smaller magnitudes than short-dated futures commodity prices. In Crosby (2005), we explain how jumps which cause parallel shifts in the term structure of (log) futures commodity prices are empirically more suitable for modelling options on gold (in this respect, gold "trades like a currency"). On the other hand, the exponentially dampened type of jumps is shown to be more suitable for modelling most other commodities (especially crude oil, natural gas and electricity).

A feature of "primary" and "daughter" commodities is that, it is observed empirically that, when there are jumps in the price of the "primary" commodity, then there are also simultaneous jumps in the price of the "daughter" commodity, albeit, generally of a different magnitude.

In this paper, we consider two commodities which we will label Commodity 1 and Commodity 2. We consider how we can adapt the Crosby (2005) model to realistically handle the case of two different commodities. Heuristically, we suppose that there are background (for example, economic) factors which influence the dynamics of futures commodity prices. These background factors are represented mathematically as Brownian motions and Poisson processes. To provide some heuristic intuition as to how the Poisson processes relate to the dynamics of futures commodity prices, we consider the following: One could imagine there being factors which caused the futures prices of both natural gas and electricity to jump simultaneously whilst there could also be factors (an outage, for example) which caused electricity prices to jump but did not cause jumps in the futures prices of natural gas. Equally there could be factors which always caused simultaneous jumps in the futures prices of crude oil and the futures prices of a refined petroleum product (although, of course, the magnitudes of the jumps could be different). At the other end of the spectrum, one could imagine modelling the futures prices of two commodities (perhaps a base metal and an energy-related commodity) which would have no simultaneous jumps at all. Of course, our aim in this paper is to price commodity derivatives for which we need to model commodity prices in the risk-neutral measure – it is not to explain price movements in the real-world physical measure. The heuristic intuition above is simply designed to provide an insight into our model.

In order to cater for all the different possible cases of modelling the futures prices of two different underlying commodities, we suppose there are M Poisson processes which drive all futures commodity prices. If, in fact, the price of a particular commodity does not jump in response to a jump of a particular Poisson process, we can cater for this by setting the jump size to be identically equal to zero.

In addition to Poisson processes, futures commodity prices are also driven by multiple Brownian motions. The diffusion volatilities associated with the Brownian motions are assumed deterministic but otherwise can be specified in a fairly flexible manner (Crosby (2005) provides more details or see Miltersen (2003) for a specification which can model seasonality in the term structure of volatilities, which is an empirically observed feature of the natural gas markets).

In this paper, we assume that interest-rates are stochastic and, therefore (Cox et al. (1981)), futures commodity prices and forward commodity prices are not the same. We will work with futures commodity prices but, results in, for example, Jamshidian (1993) and Crosby (2005) show that pricing options involving forward commodity prices is a straightforward extension.

The rest of this paper is organised as follows: In section 2, we consider a simple but realistic extension of the Crosby (2005) framework to model two underlying commodities. In section 3, we define the payoff of a simple class of exotic options. In section 4, we derive a generic formula for the price of these options using Fourier Transform methods. In section 5, we provide some numerical examples of our methodology. Section 6 is a short conclusion.

2. Extending the model to two underlying commodities

In this paper, we will make the standard assumptions that markets are frictionless and arbitrage-free. We will work exclusively in the equivalent martingale measure (EMM), under which[1] futures commodity prices are martingales, which, depending on the form of the model, may not be unique. In essence, in the case of non-uniqueness (which corresponds to market incompleteness) we assume that an EMM has been "fixed" through the market prices of standard (plain vanilla) options and by an abuse of language call this the (rather than an) EMM. Crosby (2005) provides more details. We denote expectations, at time t, with respect to the EMM by $E_t[\ ]$.

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[1] To be precise, the EMM under which futures prices are martingales is defined with respect to the money market account numeraire.

We denote the (continuously compounded) risk-free short rate, at time t, by $r(t)$ and we denote the price, at time t, of a (credit risk free) zero coupon bond maturing at time T by $P(t,T)$. We assume that interest-rates are stochastic and (see Heath et al. (1992)) follow a Gaussian interest-rate model (eg Hull-White, extended Vasicek, Babbs (1990), Hull and White (1993)), which is an arbitrage-free model consistent with any initial term structure of interest-rates. The dynamics of bond prices under the EMM are (Babbs (1990), Heath et al. (1992), Hull and White (1993)):

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sigma_P(t,T)dz_P(t),$$

where $\sigma_P(t,T)$ is a purely deterministic function of t and T, with $\sigma_P(T,T)=0$, and $dz_P(t)$ denotes standard Brownian increments. In section 5, we will provide numerical examples where we work within a one factor Gaussian (Hull-White, extended Vasicek) model in which we write

$\sigma_P(t,T) \equiv \sigma_r\left(1-\exp(-\alpha_r(T-t))\right)/\alpha_r$, where σ_r and α_r are positive constants. However, all results in this paper are extendable to any multi-factor Gaussian HJM (Heath et al. (1992)) interest-rate model without further ado.

We consider two commodities, labelled Commodity 1 and Commodity 2. We denote the futures price of Commodity i, $i=1,2$, at time t to time T (ie the futures contract, into which Commodity i, $i=1,2$, is deliverable, matures at time T) by $H_i(t,T)$. Then for each i, $i=1,2$, we assume, following Crosby (2005), that the dynamics of futures commodity prices under the EMM are:

$$\frac{dH_i(t,T)}{H_i(t-,T)} = \sum_{k=1}^{K_i}\sigma_{Hi,k}(t,T)dz_{Hi,k}(t) - \sigma_P(t,T)dz_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{i,m}\exp\left(-\int_t^T b_{i,m}(u)du\right)\right)-1\right)dN_{mt} - \sum_{m=1}^{M}e_{i,m}(t,T)dt, \tag{2.1}$$

where

$$e_{i,m}(t,T) \equiv \lambda_m(t)E_{i,m}\left(\exp\left(\gamma_{i,m}\exp\left(-\int_t^T b_{i,m}(u)du\right)\right)-1\right), \tag{2.2}$$

where, for each k, $k=1,2,...,K_i$, $\sigma_{Hi,k}(t,T)$ are purely deterministic functions of at most t and T, $dz_{Hi,k}(t)$, for each k, are standard Brownian increments (which can be correlated with each other and with $dz_P(t)$ but we assume the instantaneous correlations between these Brownian motions are constant and form a positive semi-definite correlation matrix), and N_{mt}, for each m, $m=1,...,M$, are independent Poisson processes whose intensity rates, under the EMM, at time t, are $\lambda_m(t)$ which are positive deterministic functions of at most t. The functions $b_{i,m}(t)$, for each m, and for each i, are non-negative deterministic functions which we call jump decay coefficient functions. The parameters $\gamma_{i,m}$, for each m, are parameters, which we call spot jump amplitudes. For each m, $E_{i,m}$ denotes the expectation operator, at time t, conditional on a jump occurring in N_{mt}, ie in equations 2.1 and 2.2, it computes the expected impact of jumps in the m th Poisson process on Commodity i.

We use the terminology spot jump amplitudes for the parameters $\gamma_{i,m}$ because it can be seen (Crosby (2005)) that $\gamma_{i,m}$ is the log of the jump amplitude of the futures price for a futures contract with $T=t$ (ie a futures contract for immediate delivery), ie for each m, $\gamma_{i,m}$ is the (log of the) spot jump amplitude for Commodity i. We assume that the spot jump amplitudes $\gamma_{i,m}$ are one of two possible forms, which we term those of assumption 2.1 and assumption 2.2, which in turn are linked to two possible specifications of the jump decay coefficient functions $b_{i,m}(t)$.

For each m, $m = 1,...,M$, we assume that either:

Assumption 2.1 :

The spot jump amplitudes $\gamma_{i,m}$, for each i, are assumed to be constants, which we denote by $\beta_{i,m}$. In this case, the jump decay coefficient functions $b_{i,m}(t)$ are assumed to be any non-negative deterministic function. •

Or:

Assumption 2.2 :

In this case, the jump decay coefficient functions $b_{i,m}(t)$, for each i, $i = 1,2$, are assumed to be identically equal to zero ie $b_{i,m}(t) \equiv 0$ for all t and for each i. The spot jump amplitudes $\gamma_{i,m}$ are assumed to be normally distributed random variables, with (under the EMM) mean $\beta_{i,m}$ and standard deviation $\upsilon_{i,m}$, for each i, each of which is independent of each of the Brownian motions and of each of the Poisson processes. For different m, the spot jump amplitudes $\gamma_{i,m}$ are assumed to be independent. However, for a given m, we assume that, for this m, the correlation between the spot jump amplitudes $\gamma_{1,m}$ and $\gamma_{2,m}$ is $\rho^J_{12,m}$. In other words, we assume $correl(\gamma_{1,m}, \gamma_{2,n}) \equiv \rho^J_{12,m}$ if $m = n$ and $correl(\gamma_{1,m}, \gamma_{2,n}) \equiv 0$ otherwise. We assume that, for each i and for each m, $\beta_{i,m}$, $\upsilon_{i,m}$ and $\rho^J_{12,m}$ are all constants. •

Remark 2.3 : Note that, in assumptions 2.1 and 2.2, $\beta_{1,m}$ need not equal $\beta_{2,m}$ and also that one of $\beta_{1,m}$ or $\beta_{2,m}$ may be zero (and for assumption 2.2, likewise $\upsilon_{1,m}$ and $\upsilon_{2,m}$). This allows us to capture the effect where, in response to a jump in N_{mt} at time t, the spot price $H_1(t,t)$ of Commodity 1 and the spot price $H_2(t,t)$ of Commodity 2 may jump by different magnitudes (and one may not actually jump at all).

We define the indicator functions, for each m, $m = 1,...,M$, $1_{m(2.1)} = 1$ if assumption 2.1 is satisfied, for this m, and $1_{m(2.1)} = 0$ otherwise and $1_{m(2.2)} = 1$ if assumption 2.2 is satisfied, for this m, and $1_{m(2.2)} = 0$ otherwise. Then equation 2.1 and assumptions 2.1 and 2.2 imply that

$$\sum_{m=1}^{M} e_{i,m}(t,T) = \sum_{m=1}^{M} \left(1_{m(2.1)} \lambda_m(t) \left(\exp\left(\beta_{i,m} \exp\left(-\int_t^T b_{i,m}(u) du \right) \right) - 1 \right) \right)$$

$$+ \sum_{m=1}^{M} \left(1_{m(2.2)} \lambda_m(t) \left(\exp\left(\beta_{i,m} + \frac{1}{2} \upsilon_{i,m}^2 \right) - 1 \right) \right),$$ (where we have used $b_{i,m}(t) \equiv 0$ if $1_{m(2.2)} = 1$).

Crosby (2005) provides more information about the consequences of the assumptions above and of equation 2.1. In short, the consequences are that futures commodity prices are martingales in the EMM and (with a suitable (see Crosby (2005)) form for $\sigma_{Hi,k}(t,T)$) log of the spot prices of both Commodity 1 and Commodity 2 exhibit mean reversion in the EMM. It is also shown how, when the jumps are of the type of assumption 2.1, jumps can also contribute to the effect of mean reversion and that the speed of this jump-related mean reversion is given by the values of the jump decay coefficient functions. When there are jumps, in the case of assumption 2.1 (and provided the relevant jump decay coefficient functions $b_{i,m}(t)$ are strictly positive), the prices of long-dated futures contracts jump by smaller magnitudes than short-dated futures contracts because of the exponential dampening effect of the jump decay coefficient functions in equation 2.1. This is in accordance with stylised empirical observations in most commodities markets (especially for energy-related commodities). In the case of assumption 2.2, jumps cause parallel shifts in the (log of the) futures commodity prices to all tenors because, in this case, the jump decay coefficient functions $b_{i,m}(t)$ are identically equal to zero. Stylised empirical observations suggest this is more appropriate for gold.

We have deliberately worked with very general forms of the diffusion volatility parameters $\sigma_{Hi,k}(t,T)$, the intensity rates $\lambda_m(t)$ and the jump decay coefficient functions $b_{i,m}(t)$. The specific functional forms and the values of K_1, K_2 and M would be chosen by the trader according to her intuition of the behaviour of the two underlying commodities. To help with this process, we will briefly consider possible specifications of the dynamics of the futures prices of the two commodities.

2.1 A possible specification for the jumps and the diffusion volatilities

Suppose that Commodity 1 is WTI grade crude oil. This is the "primary" commodity. It is very actively traded and there are many standard European options traded on it whose prices the trader can observe in the market. Suppose Commodity 2 is heating oil, a refined petroleum product. This is the "daughter" commodity. It is not so actively traded but there are some (but a smaller number than for WTI grade crude oil) standard European options traded on it whose prices she can observe in the market. We suppose $K_1 = 2$, $K_2 = 3$ and $M = 1$. Furthermore, we suppose the two Brownian motions driving Commodity 1 are also precisely the first two Brownian motions driving Commodity 2, with the same volatility parameters. The third Brownian motion driving Commodity 2 is specific to that commodity. More specifically, we assume

$$\sigma_{H1,1}(t,T) = \sigma_{H2,1}(t,T) = \eta_1 + \chi_1 \exp(-a_1(T-t)) \tag{2.3}$$

$$\sigma_{H1,2}(t,T) = \sigma_{H2,2}(t,T) = \chi_2 \exp(-a_2(T-t)) \tag{2.4}$$

and we assume the diffusion volatility function for the third Brownian motion (driving only Commodity 2) is of the form

$$\sigma_{H2,3}(t,T) = \chi_3 \exp(-a_3(T-t)), \tag{2.5}$$

where η_1, χ_1, χ_2, χ_3, a_1, a_2 and a_3 are all constants.

We will drop the first subscripted index for the Brownian motions in this subsection only (ie write $dz_{H1,k}(t) = dz_{H2,k}(t) \equiv dz_{H,k}(t)$, for $k = 1,2$ and $dz_{H2,k}(t) \equiv dz_{H,k}(t)$, for $k = 3$).
We define the correlations (assumed constant), for $i = 1,2,3$ and $j = 1,2,3$:

$$\rho_{i,j} \equiv correl\left(dz_{H,i}(t), dz_{H,j}(t)\right),\ \rho_{P,i} \equiv correl\left(dz_P(t), dz_{H,i}(t)\right).$$

We assume that Commodity 1 and Commodity 2 both jump in response to increments in the Poisson process N_{1t} which we assume to be of the type of assumption 2.1 and to have a constant intensity rate ie $\lambda_1(t) \equiv \lambda_1$, where λ_1 is a constant. The jump decay coefficient functions are identical for each commodity and assumed constant ie $b_{1,1}(t) \equiv b_{2,1}(t) \equiv b_1$, where b_1 is a constant. However, we assume the spot jump amplitudes are possibly different ie $\beta_{1,1}$ is not necessarily equal to $\beta_{2,1}$.

Then we can write the dynamics of Commodity 1 and Commodity 2 (under the EMM) as:

$$\frac{dH_1(t,T)}{H_1(t-,T)} = (\eta_1 + \chi_1 \exp(-a_1(T-t)))dz_{H,1}(t) + \chi_2 \exp(-a_2(T-t))dz_{H,2}(t)$$
$$- \sigma_P(t,T)dz_P(t) + \left(\exp(\beta_{1,1}\exp(-b_1(T-t))) - 1\right)dN_{1t} - e_{1,1}(t,T)dt, \tag{2.6}$$

$$\frac{dH_2(t,T)}{H_2(t-,T)} = (\eta_1 + \chi_1 \exp(-a_1(T-t)))dz_{H,1}(t) + \chi_2 \exp(-a_2(T-t))dz_{H,2}(t)$$
$$+ \chi_3 \exp(-a_3(T-t))dz_{H,3}(t)$$
$$-\sigma_P(t,T)dz_P(t) + (\exp(\beta_{2,1}\exp(-b_1(T-t))) - 1)dN_{1t} - e_{2,1}(t,T)dt. \tag{2.7}$$

If we define $R_{2/1}(t,T) \equiv \frac{H_2(t,T)}{H_1(t,T)}$ and $C_{2/1}(t) \equiv \frac{H_2(t,t)}{H_1(t,t)}$, then by Ito's lemma,

$$\frac{dR_{2/1}(t,T)}{R_{2/1}(t-,T)} = \chi_3 \exp(-a_3(T-t))dz_{H,3}(t) + \rho_{P3}\sigma_P(t,T)\chi_3 \exp(-a_3(T-t))dt$$
$$-\chi_3 \exp(-a_3(T-t))(\rho_{3,1}(\eta_1 + \chi_1 \exp(-a_1(T-t))) + \rho_{3,2}(\chi_2 \exp(-a_2(T-t))))dt$$
$$+(\exp((\beta_{2,1} - \beta_{1,1})\exp(-b_1(T-t))) - 1)dN_{1t} - (e_{2,1}(t,T) - e_{1,1}(t,T))dt. \tag{2.8}$$

Note the form of the diffusion volatility term which only depends on the Brownian increments $dz_{H,3}(t)$. In fact, utilising results in section 3 of Crosby (2005), it is now clear, from equation 2.8, that the SDE for $C_{2/1}(t)$ can be written either in the form

$$d(\ln C_{2/1}(t)) = a_3(\Lambda_J(t) - (\ln C_{2/1}(t)))dt + \chi_3 dz_{H,3}(t) + (\beta_{2,1} - \beta_{1,1})dN_{1t}, \tag{2.9}$$

or, equivalently and alternatively, in the form

$$d(\ln C_{2/1}(t)) = b_1(\Lambda_D(t) - (\ln C_{2/1}(t)))dt + \chi_3 dz_{H,3}(t) + (\beta_{2,1} - \beta_{1,1})dN_{1t}, \tag{2.10}$$

where $\Lambda_J(t)$ and $\Lambda_D(t)$ are stochastic mean reversion levels whose exact forms can easily be obtained utilising the methodology leading to proposition 3.4 of Crosby (2005), albeit at the expense of some algebra (in fact, $\Lambda_J(t)$ is a pure-jump stochastic process and $\Lambda_D(t)$ is a pure-diffusion stochastic process).

We see that the log ratio $\ln C_{2/1}(t)$ of the spot prices of the two commodities is a mean reverting stochastic process (under the EMM). This is an attractive feature for modelling, for example, the case where Commodity 1 is crude oil (the "primary" commodity) and Commodity 2 is a refined petroleum product (the "daughter" commodity) such as heating oil, because, heuristically, we would expect the price differential (and therefore also the log ratio) in the long-term to not move too far away from a long-run mean level which reflects the cost of the refining process. However, in the short-term, the log price ratio (and therefore also the arithmetic price difference) can go negative in line with the stylised empirical observations made in section 1.

We will also briefly mention how this model might be calibrated. Usually, there will be fewer actively traded options on the "daughter" commodity than on the "primary" commodity. One could estimate the parameters of the process for the "primary" commodity by calibrating to the market prices of standard options. In our example above, there would be eleven parameters, namely η_1, χ_1, χ_2, a_1, a_2, $\rho_{1,2}$, $\rho_{P,1}$, $\rho_{P,2}$, λ_1, b_1 and $\beta_{1,1}$. Having determined these eleven parameters, one could take these as given. Then one could estimate the remaining six parameters, namely χ_3, a_3, $\rho_{3,1}$, $\rho_{3,2}$, $\rho_{P,3}$, $\beta_{2,1}$, from the market prices of standard options on the "daughter" commodity. There would typically, be fewer actively traded options on the "daughter" commodity but, equally, there are fewer parameters to estimate. Of course, it would require an empirical investigation, beyond the scope of this paper, to determine how feasible our suggested calibration mechanism might be.

In order to give some intuition about the correlation between the futures prices of Commodity 1 and Commodity 2, we compute the model implied correlation between log of the futures prices of the two commodities for different tenors S and T, ie $correl(\ln(H_1(0,S)), \ln(H_2(0,T)))$, given the model specification in equations 2.6 and 2.7. The results are in figure 1. For both S and T (plotted on the x and y axes), we used the values 0, 0.5, 1, 1.5, 2, 2.5, 3 (all tenors are in years). Our parameter values are exactly as in examples 1 and 2 (see section 5). Note that when $S = T$, the correlations are lowest for the shortest tenor (ie for $S = T = 0$) but tend to one for the longest tenor ($S = T = 3$). This behaviour seems[2] quite intuitive for the case where Commodity 1 is crude oil and Commodity 2 is a refined petroleum product such as heating oil. One would expect the prices of longer-dated futures contracts on crude oil and heating oil to have a higher correlation since the difference between the (log of the) prices would reflect the (average) cost of refining (which one would expect to vary only a little). By contrast, one would expect a lower correlation between the (log of the) prices of shorter-dated futures contracts on crude oil and heating oil because the price movements would reflect additional short-term issues such as supply and demand, inventory and weather conditions.

In our example above, we have considered the dynamics of two commodities ("primary" and "daughter") where intuition suggests they will move closely (but not perfectly) together. Of course, in the case of two seemingly unconnected commodities such as, for example, natural gas and a base metal, a different specification of the jumps and the diffusion volatilities would be chosen. For example, we might consider two Poisson processes, with the first Poisson process N_{1t} only causing jumps in Commodity 1 (by having $\beta_{1,1} \neq 0$ and $\beta_{2,1} = 0$) and the second Poisson process N_{2t} only causing jumps in Commodity 2 (by having $\beta_{1,2} = 0$ and $\beta_{2,2} \neq 0$). We would also specify the diffusion terms differently. The example above is just meant for illustration.

We have illustrated how the model could be applied in a specific case of interest but, for this rest of this paper, we now return to considering the general case, as we turn our attention to pricing a class of exotic commodity options.

3. A class of exotic commodity options

Our aim is to price a European-style option whose payoff is the greater of zero and a particular function involving the futures price, at time $T_{1,1}$, of Commodity 1 deliverable at (ie the futures contract on Commodity 1 matures at) time $T_{2,1}$ and the futures price, at time $T_{1,2}$, of Commodity 2 deliverable at (ie the futures contract on Commodity 2 matures at) time $T_{2,2}$, where $T_{1,2} \leq T_{1,1}$, $T_{2,1} \geq T_{1,1}$ and $T_{2,2} \geq T_{1,2}$. The payoff is known at time $T_{1,1}$ but is paid at (a possibly later) time T_{pay}. Note $T_{pay} \geq T_{1,1} \geq T_{1,2}$.

More mathematically, we price a European-style option whose payoff is:

$$\max\left(\eta\left(\frac{H_1(T_{1,1},T_{2,1}) - K^*\left[H_2(T_{1,2},T_{2,2})\right]^{\varepsilon}}{\left[H_2(T_{1,2},T_{2,2})\right]^{\alpha}}\right),0\right), \text{ at time } T_{pay}, \tag{3.1}$$

where $\eta = 1$ if the option is a call and $\eta = -1$ if the option is a put. Note ε and α are constants and, furthermore, K^* is a constant which might, for example, account for different units of measurement. The reason for investigating options with this class of payoffs is that it contains as special cases a number of option types of interest, all of which are actively traded in the OTC commodity options markets.

We will now briefly outline (for the case of call options) some of these special cases:

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[2] Note that the parameter λ_1 is the intensity rate in the risk-neutral EMM. Clearly, this parameter could be different from the intensity rate in the real-world physical measure (in addition, had we also considered jumps of the type of assumption 2.2, the mean jump amplitudes may also be different in the two different measures). Hence, whilst the model implied (risk-neutral) correlations shown in figure 1 can provide intuition for traders, they are not directly comparable to historical correlations (implicitly evaluated in the real-world physical measure).

Spread (crack spread or dark spread or spark spread) options

These are options on the difference in price between two different underlying commodities. Their payoffs can be defined (for the "zero strike" case) via equation 3.1 with $\varepsilon = 1$ and $\alpha = 0$. In practice, we usually have $T_{1,1} = T_{1,2}$. If the underlying prices, on which the option payoff is determined, are spot prices, then we also set $T_{1,1} = T_{2,1}$ and $T_{1,2} = T_{2,2}$. •

Ratio spread or relative performance options

These are options on the ratio of the price of two different underlying commodities. Their payoffs can be defined via equation 3.1 with $\varepsilon = 1$ and $\alpha = 1$. In practice, we usually have $T_{1,1} = T_{1,2}$. If the underlying prices, on which the option payoff is determined, are spot prices, then we also set $T_{1,1} = T_{2,1}$ and $T_{1,2} = T_{2,2}$. •

Options on futures commodity price curve spreads

These are options on a single underlying physical commodity but with futures commodity contracts of different tenors ie $T_{2,1} \neq T_{2,2}$. Their payoffs can be defined via equation 3.1 with $H_1(\bullet,\bullet) \equiv H_2(\bullet,\bullet)$. Typically, we have $T_{1,1} = T_{1,2}$, $\varepsilon = 1$ and, either $\alpha = 0$ or $\alpha = 1$. •

Forward start options

These are options on a single underlying commodity in which $T_{1,2}$ is strictly less than $T_{1,1}$. The payoff is the greater of zero and the difference between the futures commodity price to a given tenor at some calendar time and the futures commodity price to the same tenor at some earlier calendar time. Their payoffs can be defined via equation 3.1 with $H_1(\bullet,\bullet) \equiv H_2(\bullet,\bullet)$, $\varepsilon = 1$ and $\alpha = 0$. For the case just described, one would have $T_{2,1} = T_{2,2}$ but other variants are possible. For example, forward start options on the spot commodity price would have $T_{1,1} = T_{2,1}$ and $T_{1,2} = T_{2,2}$. •

Ratio forward start options

Note that these options might also be called single-leg cliquets by analogy with terminology in the equity options markets. These are also options on a single underlying commodity in which $T_{1,2}$ is strictly less than $T_{1,1}$. The payoff is the greater of zero and the ratio of the futures commodity price to a given tenor at some calendar time and the futures commodity price to the same tenor at some earlier calendar time (minus a constant strike term). Their payoffs can be defined via equation 3.1 with $H_1(\bullet,\bullet) \equiv H_2(\bullet,\bullet)$, $\varepsilon = 1$ and $\alpha = 1$. Again, one could also have ratio forward start options on the spot commodity price with $T_{1,1} = T_{2,1}$ and $T_{1,2} = T_{2,2}$. •

Of course, we can also price options which are generalisations or mixtures of the special cases noted above. For example, ε and α need not be integers.

We should also make a brief comment about the time T_{pay} at which the option payoff is paid. The most common situation, in practice, is that T_{pay} would be set equal to $T_{1,1}$. However, occasionally, we observe in the OTC markets that commodity options are traded where the payoff is deferred for a short period of time after $T_{1,1}$ (and this is not just the standard two working day spot settlement but might, for example, be a period of a few weeks). For example, it might be that T_{pay} is set equal to the maturity of one of the underlying futures contracts.

We will now return, for the rest of the paper, to the completely general case of considering the class of exotic options whose payoff is given by equation 3.1.

4. Fourier Transform methodology

In this section, we will use a Fourier transform methodology, to price European-style options whose payoff is defined in equation 3.1. We will proceed along the lines of Sepp (2003) who considers the case of standard European options (on a single underlying asset).

Define, for times $t_1 \geq t$ and $t_2 \geq t$:

$$Y(t_1, T_{2,1}, t_2, T_{2,2}; t) \equiv \log\left[\left(\frac{H_1(t_1, T_{2,1})}{[H_2(t_2, T_{2,2})]^{\varepsilon}}\right) \Big/ \left(\frac{H_1(t, T_{2,1})}{[H_2(t, T_{2,2})]^{\varepsilon}}\right)\right]. \tag{4.1}$$

The price of the European-style option, whose payoff is given by equation 3.1, at time t, (for $t \leq T_{1,2} \leq T_{1,1}$) is:

$$E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right)\max\left(\eta\left(\frac{H_1(T_{1,1}, T_{2,1}) - K^*[H_2(T_{1,2}, T_{2,2})]^{\varepsilon}}{[H_2(T_{1,2}, T_{2,2})]^{\alpha}}\right), 0\right)\right]$$
$$= M_1(t) + M_2(t) - M_3(t), \tag{4.2}$$

where $$M_1(t) \equiv \frac{(1+\eta)}{2} E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right) H_1(T_{1,1}, T_{2,1})[H_2(T_{1,2}, T_{2,2})]^{-\alpha}\right] \tag{4.3}$$

and $$M_2(t) \equiv \frac{(1-\eta)}{2} E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right) K^*[H_2(T_{1,2}, T_{2,2})]^{\varepsilon-\alpha}\right] \tag{4.4}$$

and $$M_3(t) \equiv E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right)\right.$$
$$\left.[H_2(T_{1,2}, T_{2,2})]^{\varepsilon-\alpha}\min\left(\frac{H_1(t, T_{2,1})}{[H_2(t, T_{2,2})]^{\varepsilon}}\exp(Y(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t)), K^*\right)\right]. \tag{4.5}$$

The last set of equations follows from a simple algebraic arrangement.

We focus, firstly, on $M_3(t)$.

Define $f(Y(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t)) \equiv \min\left(\frac{H_1(t, T_{2,1})}{[H_2(t, T_{2,2})]^{\varepsilon}}\exp(Y(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t)), K^*\right)$

and then write

$f(Y(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t))$ in (inverse) terms of its Fourier Transform $\hat{f}(z)$ ie write

$$f(Y(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t)) = \frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty}\exp(-izY(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t))\hat{f}(z)dz, \tag{4.6}$$

where z is complex. Results in Lewis (2001) and Sepp (2003) show that, by taking the Fourier Transform of $f(Y(T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}; t))$, which exists provided $0 < z_i < 1$, where z_i is the imaginary part of z, then:

$$\hat{f}(z) = \frac{H_1(t,T_{2,1})}{[H_2(t,T_{2,2})]^{\varepsilon}} \left(\frac{1}{z^2 - iz} \right) \left(\frac{K^* [H_2(t,T_{2,2})]^{\varepsilon}}{H_1(t,T_{2,1})} \right)^{iz+1} . \quad (4.7)$$

Furthermore, by substituting equation 4.6 into equation 4.5, $M_3(t)$ is given by:

$$E_t \left[\exp\left(- \int_t^{T_{pay}} r(s)ds \right) [H_2(T_{1,2},T_{2,2})]^{\varepsilon-\alpha} \frac{1}{2\pi} \int_{iz_i-\infty}^{iz_i+\infty} \exp(-izY(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t)) \hat{f}(z) dz \right]$$

$$= \frac{1}{2\pi} \int_{iz_i-\infty}^{iz_i+\infty} E_t \left[\exp\left(- \int_t^{T_{pay}} r(s)ds \right) [H_2(T_{1,2},T_{2,2})]^{\varepsilon-\alpha} \exp(-izY(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t)) \right] \hat{f}(z) dz$$

$$\equiv \frac{1}{2\pi} \int_{iz_i-\infty}^{iz_i+\infty} \Phi(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}) \hat{f}(z) dz , \quad (4.8)$$

where $\Phi(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}) \equiv$

$$E_t \left[\exp\left(- \int_t^{T_{pay}} r(s)ds \right) [H_2(T_{1,2},T_{2,2})]^{\varepsilon-\alpha} \exp(-izY(T_{1,1},T_{2,1},T_{1,2},T_{2,2};t)) \right], \quad (4.9)$$

and where we use Fubini's theorem to justify the interchange of the integral and the expectation operator. We will call $\Phi(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2})$ the "extended" characteristic function (we have borrowed the terminology from Duffie et al. (2000) but our definition is somewhat different).

We now collect the equations above into the form of a proposition.

Proposition 4.1 : The price of the European-style option, at time t, whose payoff is defined in equation 3.1, is:

$$M_1(t) + M_2(t) - \frac{1}{2\pi} \int_{iz_i-\infty}^{iz_i+\infty} \Phi(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}) \hat{f}(z) dz . \quad (4.10)$$

Proof : From equations 4.2 and 4.8. •

Remark 4.2 : Note that equation 4.10 holds independently of the specific model for futures commodity prices. So, for example, we could consider extensions of the Crosby (2005) model which allow for, for example, stochastic volatility or alternative specifications of the jump processes (in the manner of Heston (1993), Duffie et al. (2000) and Barndorff-Nielsen and Shephard (2001)) and equation 4.10 would still be applicable, provided that the "extended" characteristic function can be calculated. Equation 4.10 (with minor modifications) could also be useful for options involving other asset classes such as equities (see Duffie et al. (2000)) or inflation (see Mercurio (2005), where it is shown that the valuation of derivatives on year-on-year inflation involves calculations very similar to valuing ratio forward start (cliquet) options). However, for the sake of brevity, we will not pursue this point further in this paper.

In the appendix, we write down the "extended" characteristic function when the dynamics of futures commodity prices are given by equation 2.1. From the form of the "extended" characteristic function, we can also easily obtain explicit forms for $M_1(t)$ and $M_2(t)$ (see the appendix).

We can now calculate the option price, via equation 4.10 provided the integral is well-defined, which requires $0 < z_i < 1$. One choice (as in Lewis (2001) and Sepp (2003)) is to evaluate the integral along the straight line given by $z = u + i/2$, where u is real.

With this choice, the price of the European-style option at time t, whose payoff is defined in equation 3.1, is:

$$M_1(t)+M_2(t)-\frac{1}{\pi}\int_0^{\infty}\Phi(-u-i/2;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2})\hat{f}(u+i/2)du\,, \tag{4.11}$$

where we have also changed the lower limit of the integration from $-\infty$ to zero by using the fact that the option price is real and hence the integrand is odd in its imaginary part and even in its real part. We will not write down the option price formula in its most explicit form as it is rather long and would not greatly enhance intuition. Equations for $M_1(t)$ and $M_2(t)$ are in the appendix and $\hat{f}(u+i/2)$ can be obtained from equation 4.7.

If the "extended" characteristic function were to be completely analytic, then it would be straightforward to evaluate the integral in equation 4.11. In particular, we can compute option prices using a single one-dimensional integration irregardless of how many Brownian motions and Poisson Processes drive the futures commodity prices. If all the Poisson processes satisfy assumption 2.2, and provided that $\int \lambda_m(s)ds$ is easily evaluated (and, of course, in practice, one would choose a form for the intensity rates $\lambda_m(s)$ so that $\int \lambda_m(s)ds$ can be evaluated analytically), then this would be the case in our model. Unfortunately, if any of the Poisson processes satisfy assumption 2.1, then our "extended" characteristic function involves integrals (see the second, third, fourth and fifth lines of equation A.2 in the appendix) which means that evaluating equation 4.11 involves at least a double integral. This is certainly computationally feasible but equally performing a double integral will be considerably slower than a single integral. Crosby (2006) shows how calculation times can be speeded up, when pricing standard (plain vanilla) European options, by using power series expansions of terms appearing in the characteristic function. A similar idea can be used here provided we make some simplifying assumptions.

As in Crosby (2006), we make the following assumption:

Assumption 4.3 : We will henceforth assume that, for each m, $m=1,\ldots,M$, $\lambda_m(s)\equiv\lambda_m$ and, for each i, $b_{i,m}(t)\equiv b_{i,m}$ are constants. Furthermore, we assume that, if for this m, the jumps satisfy assumption 2.1, then $b_{i,m}>0$. (This condition is not restrictive since if $b_{i,m}$ were to equal zero, we could treat it as in the case of assumption 2.2 which is much simpler). •

This means that we can use the power series expansions of Crosby (2006) for the terms on the second, third and fourth lines of the "extended" characteristic function (see equation A.2) (into which we would substitute $z=u+i/2$, where u is real).

In order to rapidly compute the following term (the fifth line) in equation A.2 (into which, again, we would substitute $z=u+i/2$):

$$\exp\left(\sum_{m=1}^{M}1_{m(2.1)}\int_t^{T_{1,2}}\lambda_m(s)\exp\left((\varepsilon-\alpha+iz\varepsilon)\beta_{2,m}\phi_{2,m}(s,T_{2,2})-iz\beta_{1,m}\phi_{1,m}(s,T_{2,1})\right)ds\right),$$

(where $\phi_{1,m}(s,T_{2,1})$ and $\phi_{2,m}(s,T_{2,2})$ are defined as in equation A.1 in the appendix) we will make the following additional assumption:

Assumption 4.4 : We will henceforth assume that, for each m, $b_{1,m}$ and $b_{2,m}$ are identically equal ie that $b_{1,m}\equiv b_{2,m}\equiv b_m$, say. •

Remark 4.5 : Crosby (2005) shows that futures commodity prices can be written in terms of a number of Gaussian state variables and M Poisson jump state variables. It can therefore be shown that assumption 4.4 is equivalent to saying (for assumption 2.1) that the futures prices of Commodity 1 and Commodity 2 are driven by the same jump state variables. It is shown in Crosby (2005) that our model is consistent with mean reversion, under the EMM. Not only that, but it is also shown that, when the jump processes are of the type of assumption 2.1, then jumps can also contribute to the effect of mean reversion and that the speed of this jump-related mean reversion is equal to the associated jump decay coefficient function. Hence assumption 4.4 is also equivalent to assuming that, after a jump, there is a common speed of jump-related mean reversion in Commodity 1 and Commodity 2. Although it would be an empirical matter, beyond the scope of this paper, to fully justify assumption 4.4, this assumption does, therefore, have some economic intuition. In addition, we note that assumption 4.4 is obviously a non-assumption in the special case when the option is on a single underlying commodity (see section 3, for example, options on futures commodity price curve spreads, forward start options and ratio forward start options), since it must hold.

With assumption 4.4, we can make a similar type of power series expansion which we specify in the next proposition.

Proposition 4.6 : Define $\psi_{start} \equiv \exp(-b_m(T - T_{start}))$ and $\psi_{end} \equiv \exp(-b_m(T - T_{end}))$, with $T_{start} \leq T_{end} \leq T$. Then:

$$\int_{T_{start}}^{T_{end}} \lambda_m \exp((i\omega_1 + \omega_2)\exp(-b_m(T-s)))ds = \lambda_m (T_{end} - T_{start})$$

$$+\frac{\lambda_m}{b_m}\left(\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{\left(\sqrt{\omega_1^2+\omega_2^2}\right)^n\left(\psi_{end}^n - \psi_{start}^n\right)\left(\cos(n\theta)+i\sin(n\theta)\right)}{n!}\right]\right), \tag{4.12}$$

where ω_1 and ω_2 are real numbers, independent of s , and where θ, is defined as follows:

Firstly, define $\overline{\theta}$, $0 \leq \overline{\theta} \leq \pi/2$, via $\cos\overline{\theta} \equiv \left|\omega_2 / \sqrt{\omega_1^2 + \omega_2^2}\right|$, then:

If $\omega_1 \geq 0$ and $\omega_2 \geq 0$, then $\theta = \overline{\theta}$, else if $\omega_1 \geq 0$ and $\omega_2 < 0$, then $\theta = \pi - \overline{\theta}$, else if $\omega_1 < 0$ and $\omega_2 \geq 0$, then $\theta = 2\pi - \overline{\theta}$, else if $\omega_1 < 0$ and $\omega_2 < 0$, then $\theta = \pi + \overline{\theta}$.

Proof: This proposition is just a generalisation of proposition 3.3 in Crosby (2006) and can be proved in an identical fashion. Therefore, the proof is omitted[3]. •

All the integrals (see the second, third, fourth and fifth lines of equation A.2) which appear in the "extended" characteristic function can be nested in a form which enables them to be evaluated by proposition 4.6, provided assumptions 4.3 and 4.4 hold. Hence, we can quickly and easily evaluate the "extended" characteristic function. We can also evaluate $M_1(t)$ and $M_2(t)$ (see appendix) in the same way. We can then very rapidly, using standard one dimensional numerical integration techniques, compute the integral in equation 4.11 and hence also compute the price of the European-style option whose payoff is defined in equation 3.1.

5. Numerical examples and results

In this section, we will provide four numerical examples, labelled examples 1, 2, 3 and 4, of our methodology, the results of which are in tables 1, 2, 3 and 4 respectively. In all four examples, we value European-style options, whose payoff is defined in equation 3.1, using equation 4.11.

We evaluate the integral with respect to u in equation 4.11 using Simpson's rule with 1024 points. Examining the forms of equations A.2 and 4.11, we see that for large u, the integrand behaves

[3] It is straightforward to see that the power-series expansion in equation 4.12 will be rapidly convergent. Indeed the modulus of the term appearing in the square brackets is guaranteed to be monotonically declining to zero when $n > \max\left(2, \sqrt{\omega_1^2 + \omega_2^2}\right)$.

asymptotically like $\exp\left(-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\Sigma^2\left(t,T_{1,1},T_{1,2},T_{2,1},T_{2,2}\right)\right)\Big/\left(u^2+\frac{1}{4}\right)$ which clearly tends to zero rapidly as $u \to \infty$, since $\Sigma^2\left(t,T_{1,1},T_{1,2},T_{2,1},T_{2,2}\right) \geq 0$. We truncate the upper limit of the integral when the value of u is such $\exp\left(-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\Sigma^2\left(t,T_{1,1},T_{1,2},T_{2,1},T_{2,2}\right)\right)\Big/\left(u^2+\frac{1}{4}\right)$ is less than 10^{-8}. We truncate the infinite series in equation 4.12 when the value of an additional term in the series has converged to less than 10^{-12}.

In all four examples, we assume that the initial (ie as of the valuation date of the options that we will value in our examples) futures prices of Commodity 1 to all maturities are 40 and the initial futures prices of Commodity 2 to all maturities are 41. We assume that the initial interest-rate yield curve is flat with a continuously compounded risk-free rate of 0.044 ie we assume that the discount factors for all maturities T, as of the valuation date, time t, of the options that we will value, are all of the form $\exp(-0.044(T-t))$. Interest-rates are stochastic and evolve following a one factor Hull-White (extended Vasicek) model in which
$\sigma_P(t,T) \equiv \sigma_r\left(1-\exp(-\alpha_r(T-t))\right)/\alpha_r$, where $\sigma_r = 0.012$ and $\alpha_r = 0.125$.

In all four examples, we use the same form for the diffusion parameters as in equations 2.3 to 2.5 in section 2.1. That is, we suppose $K_1 = 2$ and $K_2 = 3$ and, furthermore, we suppose
$\eta_1 = 0.12$, $\chi_1 = 0.22$, $\chi_2 = 0.25$, $\chi_3 = 0.242$, $a_1 = 0.9$, $a_2 = 0.7$, $a_3 = 1.5$.
We assume all correlations are 0.05 ie for all $i = 1,2,3$ and $j = 1,2,3$:
$\rho_{i,j} = 0.05$ and $\rho_{P,i} = 0.05$. Note that all these parameters are just for illustration.

In all four examples, we assume that the maturities of the futures contracts on Commodity 1 are of the form $T_{2,1} = T_{1,1} + (31/365)$ and on Commodity 2 of the form $T_{2,2} = T_{1,2} + (91/365)$ ie the futures contracts on Commodity 1 and Commodity 2 mature 31 days and 91 days respectively after $T_{1,1}$ and $T_{1,2}$. In all four examples, we set $T_{pay} \equiv T_{1,1}$. We specify $T_{1,1}$ and $T_{1,2}$ in the examples.

In each example, we value six options and all of them are calls (ie $\eta = 1$). For the first three options, $\alpha = 0$, $\varepsilon = 1$ and the values of K^* are 0.95, 0.975 and 1. The fourth, fifth and sixth options have $\alpha = 1$ and, again, $\varepsilon = 1$ and the values of K^* are 0.95, 0.975 and 1. Thus, we evaluate spread options for three different values of K^* and ratio spread options for the same three values of K^* in each example.

Example 1 :

In example 1, we assume $T_{1,1} = 1$ and $T_{1,2} = 1$. We assume that there is one Poisson process, $M = 1$, and it satisfies assumption 2.1 and it has an intensity rate $\lambda_1 = 0.512$. As in the example in section 2.1, both Commodity 1 and Commodity 2 exhibit jumps of non-zero magnitude in response to jumps in this Poisson process. We assume $b_{1,1} = b_{2,1} = 1.55$, $\beta_{1,1} = 0.55$, $\beta_{2,1} = 0.35$. We price the six different options and the results are in table 1. •

Example 2 :

In example 2, we assume $T_{1,1} = 3$ and $T_{1,2} = 2$. This means that the maturities of the futures contracts on Commodity 1 and Commodity 2 are approximately 3.08493151 and 2.24931507 years respectively. Note that because $T_{1,2}$ is strictly less than $T_{1,1}$, the options in this example can also be viewed as hybrid forward start options (for the first three options where $\alpha = 0$) and ratio forward start options (for the fourth, fifth and sixth options where $\alpha = 1$) involving two different commodities. We assume that all the jump parameters are exactly the same as in example 1. We price the six different options and the results are in table 2. •

Example 3 :

In example 3, we assume $T_{1,1} = 1$ and $T_{1,2} = 1$. We use exactly the same diffusion parameters as in examples 1 and 2 but to provide a contrast with those examples, we assume that there are two Poisson process, $M = 2$, and they both satisfy assumption 2.1 and they have intensity rates $\lambda_1 = 0.512$ and $\lambda_2 = 0.47$ respectively. Commodity 1 jumps but Commodity 2 does not jump in responses to jumps in this first Poisson process N_{1t}. Conversely, Commodity 2 jumps but Commodity 1 does not jump in responses to jumps in this second Poisson process N_{2t}. We assume $b_{1,1} = b_{2,1} = 1.55$, $b_{1,2} = b_{2,2} = 1.55$, $\beta_{1,1} = 0.55$, $\beta_{2,1} = 0$, $\beta_{1,2} = 0$, $\beta_{2,2} = 0.35$. We price the six different options and the results are in table 3. •

Example 4 :

In example 4, we assume $T_{1,1} = 3$ and $T_{1,2} = 2$. We assume that there are two Poisson processes again and that all the jump parameters are exactly the same as in example 3. We assume that all the diffusion parameters are exactly the same as in examples 1, 2 and 3. Note that, as in example 2, because $T_{1,2}$ is strictly less than $T_{1,1}$, the options in this example can also be viewed as hybrid forward start and ratio forward start options. We price the six different options and the results are in table 4. •

Computations were performed on a desk-top p.c., running at 2.8 GHz, with Microsoft Windows XP Professional, with 1 Gb of RAM with a program written in Microsoft C++. The total calculation time for all 24 options in examples 1 to 4 was 0.532 seconds or an average of less than 23 milliseconds per option. By significantly increasing the number of points in the numerical integration and by significantly reducing the tolerances used to truncate the upper limit of the integral (in equation 4.11) and the power series expansions (as in equation 4.12), we were able to confirm that in proportional (ie proportional to the calculated option prices) terms, all the option prices in tables 1 to 4 are accurate to at least one part in 500,000 and, also, that in absolute terms, all the option prices are accurate to at least 5 decimal places. So our algorithm is both fast and accurate.

Note how the option prices in examples 3 and 4 are higher than the corresponding option prices in examples 1 and 2 respectively. This is intuitive given the different specifications of the jump processes driving futures commodity prices, between, on the one hand, examples 1 and 2, and, on the other hand, examples 3 and 4, and given the arguments we presented after equation 2.10.

6. Conclusions

We have extended the Crosby (2005) model to simultaneously model the prices of multiple commodities. We then priced a class of simple exotic options which includes those whose payoffs involve two different underlying commodities, or a single underlying commodity but with futures contracts of two different tenors or the price of a single underlying futures contract observed at two different calendar times. This class of exotic options includes common exotics such as (crack, dark or spark) spread options, ratio spread options, forward start options and ratio forward start options (single leg cliquets). We have shown that these exotic options can be priced using Fourier methods in any model in which the relevant "extended" characteristic function is known analytically or can be computed rapidly. The Crosby (2005) model falls into the latter category. We have provided some numerical examples which demonstrate that our methodology is both fast and accurate.

Finally, we will briefly mention two possible areas for future research:

(i) We have focussed, when pricing spread options in this paper, on the "zero strike" case. Dempster and Hong (2000) show how "non-zero-strike" spread options can be priced using a two-dimensional Fast Fourier Transform methodology combined with an ingenious decomposition of the option payoff analogous to Riemann sums. Their approach (combined with assumptions 4.3 and 4.4 and the power series expansion of proposition 4.6) could be used to price "non-zero-strike" spread options within the framework of this paper. It might also be possible to extend the Dempster and Hong (2000) approach in order to price more exotic variations of some of the option types we discussed in section 3.

(ii) In section 2.1, we provided an example of specifying the dynamics of the futures prices of two different commodities based on heuristics and trader-intuition. It might be possible to construct a more systematic approach based on suitable extensions of the methodology described in section 3 of Casassus and Collin-Dufresne (2005). However, we leave this for future research.

Appendix

In order to obtain the forms for $M_1(t)$ and $M_2(t)$, defined in equations 4.3 and 4.4, we can essentially use the "extended" characteristic function (defined in equation 4.9 and given explicitly in equation A.2 below), into which we substitute $z = i$ and $z = 0$ respectively, then :

$$
\begin{aligned}
M_1(t) &\equiv \frac{(1+\eta)}{2} E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right) H_1(T_{1,1}, T_{2,1})[H_2(T_{1,2}, T_{2,2})]^{-\alpha}\right] \\
&= \frac{(1+\eta)}{2} \frac{H_1(t, T_{2,1})}{[H_2(t, T_{2,2})]^{\varepsilon}} \Phi(-i; t, T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2})
\end{aligned}
$$

and

$$
\begin{aligned}
M_2(t) &\equiv \frac{(1-\eta)}{2} E_t\left[\exp\left(-\int_t^{T_{pay}} r(s)ds\right) K^*[H_2(T_{1,2}, T_{2,2})]^{\varepsilon-\alpha}\right] \\
&= \frac{(1-\eta)K^*}{2} \Phi(0; t, T_{1,1}, T_{2,1}, T_{1,2}, T_{2,2}).
\end{aligned}
$$

We will now proceed to write down the "extended" characteristic function when the dynamics of futures commodity prices are given by equation 2.1, after defining the following notation:

For each i, $i = 1,2$, $\frac{dH_i^C(t,T)}{H_i^C(t,T)} \equiv \sum_{k=1}^{K_i} \sigma_{Hi,k}(t,T)dz_{Hi,k}(t) - \sigma_P(t,T)dz_P(t)$, ie $\frac{dH_i^C(t,T)}{H_i^C(t,T)}$ denotes the purely continuous martingale component in the SDE for Commodity i.

For each i, $i = 1,2$, $\phi_{i,m}(s,T) \equiv \exp\left(-\int_s^T b_{i,m}(u)du\right)$. (A.1)

Define
$$
\begin{aligned}
U(t, T_{1,2}, T_{2,2}) \equiv &-(\varepsilon - \alpha)\int_t^{T_{1,2}} \mathrm{cov}\left(\frac{dP(s, T_{pay})}{P(s, T_{pay})}, \frac{dH_2^C(s, T_{2,2})}{H_2^C(s, T_{2,2})}\right)ds \\
&-\frac{1}{2}(\varepsilon - \alpha)(\varepsilon - \alpha - 1)\int_t^{T_{1,2}} \mathrm{var}\left(\frac{dH_2^C(s, T_{2,2})}{H_2^C(s, T_{2,2})}\right)ds.
\end{aligned}
$$

Define $W(t, T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}) \equiv$

$$
\begin{aligned}
&\int_t^{T_{1,1}} \mathrm{cov}\left(\frac{dP(s, T_{pay})}{P(s, T_{pay})}, \frac{dH_1^C(s, T_{2,1})}{H_1^C(s, T_{2,1})}\right)ds - \varepsilon\int_t^{T_{1,2}} \mathrm{cov}\left(\frac{dP(s, T_{pay})}{P(s, T_{pay})}, \frac{dH_2^C(s, T_{2,2})}{H_2^C(s, T_{2,2})}\right)ds \\
&-\alpha\int_t^{T_{1,2}} \mathrm{cov}\left(\frac{dH_1^C(s, T_{2,1})}{H_1^C(s, T_{2,1})}, \frac{dH_2^C(s, T_{2,2})}{H_2^C(s, T_{2,2})}\right)ds + \frac{1}{2}\left(\varepsilon + 2\varepsilon\alpha - \varepsilon^2\right)\int_t^{T_{1,2}} \mathrm{var}\left(\frac{dH_2^C(s, T_{2,2})}{H_2^C(s, T_{2,2})}\right)ds.
\end{aligned}
$$

Define $\Sigma^2\left(t,T_{1,1},T_{1,2},T_{2,1},T_{2,2}\right)\equiv\int_t^{T_{1,1}}\operatorname{var}\left(\frac{dH_1^C\left(s,T_{2,1}\right)}{H_1^C\left(s,T_{2,1}\right)}\right)ds$

$$-2\varepsilon\int_t^{T_{1,2}}\operatorname{cov}\left(\frac{dH_1^C\left(s,T_{2,1}\right)}{H_1^C\left(s,T_{2,1}\right)},\frac{dH_2^C\left(s,T_{2,2}\right)}{H_2^C\left(s,T_{2,2}\right)}\right)ds+\varepsilon^2\int_t^{T_{1,2}}\operatorname{var}\left(\frac{dH_2^C\left(s,T_{2,2}\right)}{H_2^C\left(s,T_{2,2}\right)}\right)ds.$$

In order to compute the "extended" characteristic function, we will use the fact that Brownian motions and Poisson processes have independent increments. Then by direct calculation:

$$\begin{aligned}
&\Phi\left(-z;t,T_{1,1},T_{2,1},T_{1,2},T_{2,2}\right)=\exp\left(-\sum_{m=1}^{M}\int_t^{T_{1,1}}\lambda_m(s)ds\right)\\
&\exp\left(\sum_{m=1}^{M}1_{m(2.1)}\int_{T_{1,2}}^{T_{1,1}}\lambda_m(s)\exp\left(-iz\beta_{1,m}\phi_{1,m}\left(s,T_{2,1}\right)\right)ds\right)\\
&\exp\left(iz\sum_{m=1}^{M}1_{m(2.1)}\int_t^{T_{1,1}}\lambda_m(s)\left(\exp\left(\beta_{1,m}\phi_{1,m}\left(s,T_{2,1}\right)\right)-1\right)ds\right)\\
&\exp\left(-\left(\varepsilon-\alpha+iz\varepsilon\right)\sum_{m=1}^{M}1_{m(2.1)}\int_t^{T_{1,2}}\lambda_m(s)\left(\exp\left(\beta_{2,m}\phi_{2,m}\left(s,T_{2,2}\right)\right)-1\right)ds\right)\\
&\exp\left(\sum_{m=1}^{M}1_{m(2.1)}\int_t^{T_{1,2}}\lambda_m(s)\exp\left(\left(\varepsilon-\alpha+iz\varepsilon\right)\beta_{2,m}\phi_{2,m}\left(s,T_{2,2}\right)-iz\beta_{1,m}\phi_{1,m}\left(s,T_{2,1}\right)\right)ds\right)\\
&\exp\left(\sum_{m=1}^{M}1_{m(2.2)}\left(\int_{T_{1,2}}^{T_{1,1}}\lambda_m(s)\right)\exp\left(-iz\beta_{1,m}-\frac{1}{2}\upsilon_{1,m}^2z^2\right)\right)\\
&\exp\left(iz\sum_{m=1}^{M}1_{m(2.2)}\left(\int_t^{T_{1,1}}\lambda_m(s)ds\right)\left(\exp\left(\beta_{1,m}+\frac{1}{2}\upsilon_{1,m}^2\right)-1\right)\right)\\
&\exp\left(-\left(\varepsilon-\alpha+iz\varepsilon\right)\sum_{m=1}^{M}1_{m(2.2)}\left(\int_t^{T_{1,2}}\lambda_m(s)ds\right)\left(\exp\left(\beta_{2,m}+\frac{1}{2}\upsilon_{2,m}^2\right)-1\right)\right)\\
&\exp\left(\sum_{m=1}^{M}1_{m(2.2)}\left(\int_t^{T_{1,2}}\lambda_m(s)ds\right)\exp\left\{\left(\varepsilon-\alpha+iz\varepsilon\right)\beta_{2,m}+\frac{1}{2}\left(\varepsilon-\alpha+iz\varepsilon\right)^2\upsilon_{2,m}^2\right.\right.\\
&\qquad\left.\left.-iz\beta_{1,m}-\frac{1}{2}\upsilon_{1,m}^2z^2-\left(\varepsilon-\alpha+iz\varepsilon\right)iz\rho_{12,m}^J\upsilon_{1,m}\upsilon_{2,m}\right\}\right)\\
&\exp\left(\frac{1}{2}\left(iz-z^2\right)\Sigma^2\left(t,T_{1,1},T_{1,2},T_{2,1},T_{2,2}\right)\right)\exp\left(-izW\left(t,T_{1,1},T_{1,2},T_{2,1},T_{2,2}\right)\right)\\
&P\left(t,T_{pay}\right)\exp\left(-U\left(t,T_{1,2},T_{2,2}\right)\right)\left(H_2\left(t,T_{2,2}\right)\right)^{\varepsilon-\alpha}.
\end{aligned}\tag{A.2}$$

Figure 1 :

Graph of the model implied instantaneous correlation $correl(\ln(H_1(0,S)),\ln(H_2(0,T)))$ between the futures prices of Commodity 1 and Commodity 2, for different tenors S and T, given the model specification in equations 2.6 and 2.7. For both S and T (plotted on the x and y axes), we used the values 0, 0.5, 1, 1.5, 2, 2.5, 3 (all tenors are in years). Our parameter values are as in examples 1 and 2 (see section 5).

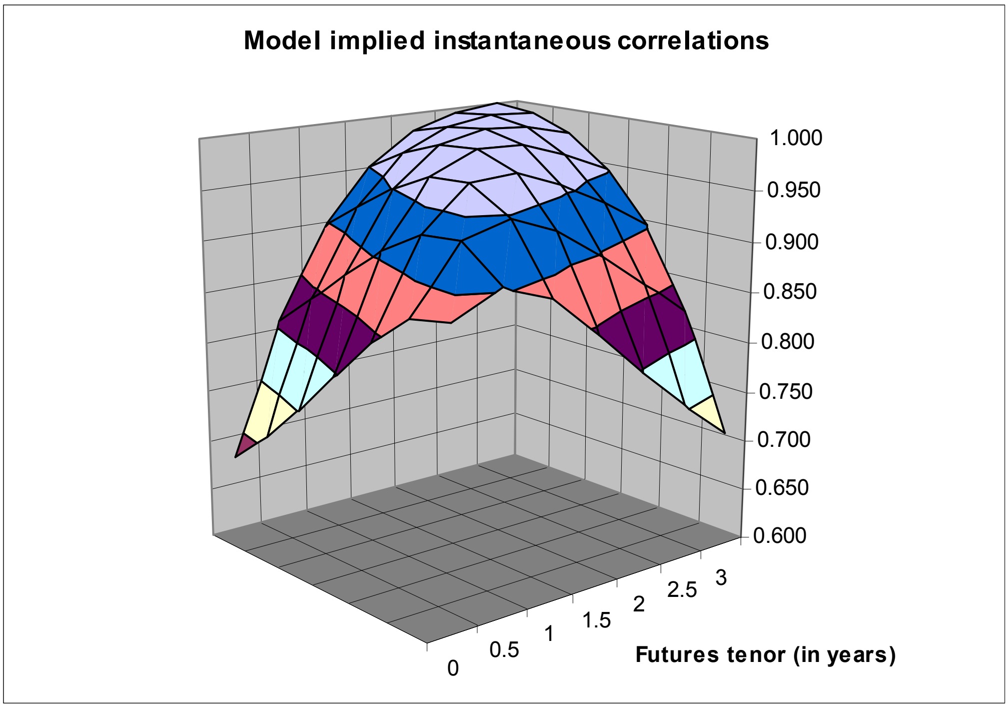

Table 1 :

There is one Poisson process. $T_{1,1} = 1$, $T_{1,2} = 1$, $T_{2,1} = 1 + (31/365)$, $T_{2,2} = 1 + (91/365)$.
The values of K^* are across the first row and the option prices are in bold across the second row.

$\alpha = 0$, $\varepsilon = 1$ (spread options)			$\alpha = 1$, $\varepsilon = 1$ (ratio spread options)		
0.95	0.975	1.0	0.95	0.975	1.0
2.64579	**2.19204**	**1.80901**	**0.05799**	**0.04737**	**0.03852**

Table 2 :

There is one Poisson process. $T_{1,1} = 3$, $T_{1,2} = 2$, $T_{2,1} = 3 + (31/365)$, $T_{2,2} = 2 + (91/365)$.
The values of K^* are across the first row and the option prices are in bold across the second row.

$\alpha = 0$, $\varepsilon = 1$ (spread options)			$\alpha = 1$, $\varepsilon = 1$ (ratio spread options)		
0.95	0.975	1.0	0.95	0.975	1.0
6.04522	**5.66903**	**5.31508**	**0.17500**	**0.16471**	**0.15498**

Table 3 :

There are two Poisson processes. $T_{1,1} = 1$, $T_{1,2} = 1$, $T_{2,1} = 1 + (31/365)$, $T_{2,2} = 1 + (91/365)$.
The values of K^* are across the first row and the option prices are in bold across the second row.

$\alpha = 0$, $\varepsilon = 1$ (spread options)			$\alpha = 1$, $\varepsilon = 1$ (ratio spread options)		
0.95	0.975	1.0	0.95	0.975	1.0
4.02340	**3.63361**	**3.28715**	**0.10248**	**0.09258**	**0.08379**

Table 4 :

There are two Poisson processes. $T_{1,1} = 3$, $T_{1,2} = 2$, $T_{2,1} = 3 + (31/365)$, $T_{2,2} = 2 + (91/365)$.
The values of K^* are across the first row and the option prices are in bold across the second row.

$\alpha = 0$, $\varepsilon = 1$ (spread options)			$\alpha = 1$, $\varepsilon = 1$ (ratio spread options)		
0.95	0.975	1.0	0.95	0.975	1.0
6.17001	**5.79409**	**5.43994**	**0.17959**	**0.16926**	**0.15949**

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