Introduction to jump and Lévy processes

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- The aim of this lecture is to give you a basic introduction to jump and Lévy processes and their uses in mathematical finance.

- Processes with jumps are being more commonly used. They are especially useful for modelling credit events (jumps to default) and commodity prices - although given the extreme market moves of Autumn 2008, they may well become more common in other asset classes as well.

- We know Brownian motion has continuous sample paths. Intuitively, large moves happen very rarely. Jump processes have discontinuous sample paths and, therefore, they allow for large sudden moves in the underlying price process. They can also capture skewness and excess kurtosis in price returns. The realised variance of a jump process is stochastic (even without introducing any notion of stochastic volatility). These properties are all observed in historical time-series (generated under the real-world measure).

- Additionally, jump processes (modelled under the (or a) equivalent martingale measure (the measure is rarely unique for jump processes because the market is typically incomplete)) can capture the effect of volatility smiles and skews which makes them attractive for derivatives pricing.

Motivation 2

- There are barriers to using jump processes.
- The mathematical machinery is a little more complex.
- Instead of needing to solve a partial differential equation (PDE) to price derivatives, one has to solve a partial-integro differential equation (PIDE) which is much harder.
- On the positive side, pricing vanilla options by Fourier methods is quite straightforward and there are efficient ways of performing Monte Carlo simulations.

- We will freely use three important pieces of maths and one definition.

- The characteristic function of a process X_t is defined to be: $\mathbb{E}[\exp(izX_t))]$. Here z can certainly be any real number and it can also, potentially, be complex as long as the expectation remains finite (which may place a restriction on its imaginary part).

- A probability distribution, with characteristic function $\phi(z)$ is said to be infinitely divisible if, for every integer $n \geq 1$, $\phi(z)$ is also the n^{th} power of a characteristic function. Not all probability distributions satisfy this (eg. uniform $U(0,1)$) but many do (eg. Gaussian, Poisson, gamma).

- A process X_t, defined for $t \geq 0$, has stationary increments if the distribution of $X_{t+s} - X_t$, for any $t \geq 0$, depends only on s and not upon t, for all $s \geq 0$.

- A process X_t, defined for $t \geq 0$, has independent increments if, for every possible set of times $0 \leq t_1 < t_2 \leq t_3 < t_4$, $X_{t_4} - X_{t_3}$ is independent of $X_{t_2} - X_{t_1}$.

- The prototype jump process is the Poisson process $N(t)$. It is a process with stationary and independent increments. It starts at time $t_0 \equiv 0$ with $N(t_0) = 0$. It has Poisson distributed increments which means its support (i.e. the set of values it can take) lies on the non-negative integers. We denote by λ the intensity rate of the process (with λ constant, $0 < \lambda < \infty$). Then the probability that $N(t) = n$ is given by:

$$\mathrm{Prob}(N(t) = n) = \frac{\exp(-\lambda t)(\lambda t)^n}{n!}.$$

- Note that:

$$\sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} = 1,$$

(from the power series representation of the exp function) which is required for a distribution function.

- We can compute the characteristic function of the Poisson process:

- Indeed:

$$
\begin{aligned}
\mathbb{E}_{t_0}[\exp(izN(t))] &= \mathbb{E}_{t_0}[\mathbb{E}_{t_0}[\exp(izN(t))]|N(t)=n]] = \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \exp(izn) \\
&= \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t e^{iz})^n}{n!} = \exp(-\lambda t)\exp(\lambda t e^{iz}) \\
&= \exp(\lambda t(e^{iz}-1)).
\end{aligned}
$$

  We can differentiate the above and set $z = 0$, to find:

$$
\mathbb{E}_{t_0}[N(t)] = \lambda t.
$$

- We see that the intensity rate λ is the expected number of jumps per unit of time.

- Before moving on, we state an important result concerning multiple Poisson processes. Suppose we have K independent Poisson processes $N_k(t)$ with intensity rates λ_k, $k = 1, ..., \ldots, K$ $(0 < \lambda_k < \infty)$. We consider the process $L(t)$, which is the sum of these processes, i.e. $L(t) = \sum_{k=1}^{K} N_k(t)$, then: $L(t)$ is a Poisson process with intensity rate $\sum_{k=1}^{K} \lambda_k$. Proof: A probabilistic proof is in Ross (1997). Alternatively, note that:

$$\begin{aligned}
\mathbb{E}_{t_0}[\exp(izL(t))] &= \mathbb{E}_{t_0}[\exp(iz\sum_{k=1}^{K} N_k(t))] = \prod_{k=1}^{K} \mathbb{E}_{t_0}[\exp(izN_k(t))] \text{ (independence)} \\
&= \prod_{k=1}^{K}[\exp(\lambda_k t(e^{iz} - 1))] = \exp(\sum_{k=1}^{K} \lambda_k t(e^{iz} - 1)),
\end{aligned}$$

  which is the characteristic function of a Poisson process with intensity rate $\sum_{k=1}^{K} \lambda_k$.

- We use this result later.

- The simple Poisson process defined above is a starting point, but it can only jump up by unity, so we need something richer to have a suitable model for modelling risk-neutral stock prices.

- Firstly, we would like to have a range of possible (and, indeed, even random) jump amplitudes (not just unity). We can capture this by using a compound Poisson process. Secondly, we have the intuition that jumps are rare events. Most of the time, we expect stock prices movements to be small. We can capture this by adding a Brownian motion to the compound Poisson process. The sum is called a “jump-diffusion process”.

- The compound Poisson process is constructed as follows: We have a Poisson process $N(t)$ (with $N(t_0) = 0$) with intensity rate λ (with λ constant, $0 < \lambda < \infty$). Every time the Poisson process jumps, we draw an independent and identically distributed random variable J (which, as a special case, could be a constant, not necessarily one - nor necessarily positive) from a given distribution for which the probability of J being in the (Borel) set A is $\mathrm{Prob}(J \in A) = \nu(A)/\lambda$, further implying $\nu(\mathbb{R}) = \lambda$ which can also be expressed as $\lambda = \int_{-\infty}^{\infty} \nu(dx)$.

- We construct a process X_t, with $X_{t_0} = 0$, as a sum of a Brownian motion W_t (with $W_{t_0} = 0$) with volatility term σ $(0 < \sigma < \infty)$ and the compound Poisson process, together with a drift term γ.

- We write (recall $t_0 \equiv 0$):

$$X_t = \gamma t + \sigma W_t + \sum_{k=1}^{N(t)} J.$$

Note that $N(t)$ is the (random) number of jumps until time t and that J is (in general) a random variable - more specifically, it is the outcome of an independent draw from the specified distribution - so it can (and will) be different for each k. The convention is always that, if $N(t) = 0$, i.e. the sum is empty, then the sum is set to zero.

- W_t, $N(t)$ and J are independent.

- We compute the characteristic function of X_t. The characteristic function of the Brownian component is $\mathbb{E}_{t_0}[\exp(iz\sigma W_t)] = \exp(-\frac{1}{2}\sigma^2 z^2 t)$ (standard result or integrate over the Gaussian density function and complete the square).

- Hence, the characteristic function of X_t is (using independence of $N(t)$ and J and then that the random variables J are independent and identically distributed and then the power series for the exp function):

- $$\begin{aligned}
\mathbb{E}_{t_0}[\exp(iz(X_t))] &= \mathbb{E}_{t_0}[\exp(iz(\gamma t + \sigma W_t + \sum_{n=1}^{N(t)} J))] \\
&= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t)\mathbb{E}_{t_0}[\mathbb{E}_{t_0}[\exp(iz(\sum_{n=1}^{N(t)} J))|J = x, N(t) = k]] \\
&= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t) \sum_{k=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^k}{k!} \prod_{n=1}^{k} (\int_{-\infty}^{\infty} e^{izx} \frac{\nu(dx)}{\lambda}) \\
&= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t) \sum_{k=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t \int_{-\infty}^{\infty} e^{izx} \frac{\nu(dx)}{\lambda})^k}{k!} \\
&= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t) \exp(-\lambda t) \exp(\lambda t \int_{-\infty}^{\infty} e^{izx} \frac{\nu(dx)}{\lambda}) \\
&= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t) \exp(\lambda t(\int_{-\infty}^{\infty} e^{izx} \frac{\nu(dx)}{\lambda} - 1)).
\end{aligned}$$

- This can be re-expressed as:

$$\begin{aligned}\mathbb{E}_{t_0}[\exp(iz(X_t))] &= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t)\exp(\lambda t(\int_{-\infty}^{\infty} e^{izx}\frac{\nu(dx)}{\lambda} - 1)) \\ &= \exp(iz\gamma t - \frac{1}{2}\sigma^2 z^2 t)\exp(t(\int_{-\infty}^{\infty}(e^{izx} - 1)\nu(dx))).\end{aligned}$$

- The last two lines are equivalent since $\lambda = \int_{-\infty}^{\infty} \nu(dx)$.

- The characteristic function of X_t can also be expressed as $\mathbb{E}_{t_0}[\exp(iz(X_t))] \equiv \exp(t\Phi(z))$, where $\Phi(z)$ is called the "characteristic exponent":

$$\Phi(z) \equiv iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty}(e^{izx} - 1)\nu(dx).$$

- We note one fact for future reference:
- Denote the expectation of J by m_J i.e. $m_J = \int_{-\infty}^{\infty} x \frac{\nu(dx)}{\lambda}$. If we form the process $M_t = X_t - \lambda m_J t$, then $M_{t_0} = 0$ and, furthermore, the characteristic function of M_t is:

$$\mathbb{E}_{t_0}[\exp(iz(M_t))] = \exp(t(iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx)\nu(dx))).$$

  This means the characteristic exponent of M_t is:

$$iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx)\nu(dx).$$

- We will be concerned with constructing a model for the evolution of stock prices in a risk-neutral world. We assume the absence of arbitrage which certainly guarantees the existence of at least one equivalent martingale measure. However, for models with jumps, because the market is (except in rare special cases) incomplete, the equivalent martingale measure is not unique. That is, multiple equivalent martingale measures exist, all of which are consistent with no-arbitrage but the price of a derivative security will be different for each possible equivalent martingale measure.

- We will assume that one such martingale measure $\mathbb{Q}$ has been fixed (typically by a calibration to the market prices of options) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0 \equiv 0}, \mathbb{Q})$. The filtration $\mathcal{F}$ is the natural filtration generated by W_t, $N(t)$ and J.

- The jump-diffusion process introduced above is rich enough to be used to model the dynamics of stock prices under $\mathbb{Q}$. As with the standard Black-Scholes model, we model log prices.

- We assume that the stock price, $S(t)$, at time t, with $t \geq t_0 \equiv 0$, $X_{t_0} = 0$, is:

$$\begin{aligned} S(t) &= S(t_0)\exp((r-q)t)\exp(X_t) \\ &= S(t_0)\exp((r-q)t)\exp(\gamma t + \sigma W_t + \sum_{n=1}^{N(t)} J), \end{aligned}$$

  where r is the risk-free interest-rate and q is the dividend yield (both assumed constant for notational simplicity).

- However, we must specify γ. We know that the drift rate on the stock, under $\mathbb{Q}$, must be $r-q$. Equivalently, $\mathbb{E}_{t_1}^{\mathbb{Q}}[S(t_2)] = S(t_1)\exp((r-q)(t_2-t_1))$, for any $t_2 \geq t_1 \geq t_0 \equiv 0$. Therefore, we must choose γ such that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$ to have a model consistent with no-arbitrage.

- We must choose:

$$\gamma = -\frac{1}{2}\sigma^2 - \int_{-\infty}^{\infty} (e^x - 1)\nu(dx).$$

- Why this choice? The characteristic function becomes:

$$\begin{aligned}\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iz(X_t))] \;&=\; \exp(iz\left(-\frac{1}{2}\sigma^2 - \int_{-\infty}^{\infty}(e^x - 1)\nu(dx)\right) t - \frac{1}{2}\sigma^2 z^2 t) \\ &\;\; \exp(t(\int_{-\infty}^{\infty}(e^{izx} - 1)\nu(dx))).\end{aligned}$$

Now set $z = -i$ or $iz = 1$, and it is clear, that with this choice of γ, that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$.

- Rearranging, we can now write the stock price dynamics as:

$$\log S(t) = \log S(t_0) + (r - q - \frac{1}{2}\sigma^2 - k)t + \sigma W_t + \sum_{n=1}^{N(t)} J,$$

  where $k \equiv \int_{-\infty}^{\infty} (e^x - 1)\nu(dx)$.

- Note that the term $-\frac{1}{2}\sigma^2$ is the usual term we see in the standard Black-Scholes (pure diffusion) model.

- In differential notation, we write this as:

$$d \log S(t) = (r - q - \frac{1}{2}\sigma^2 - k)dt + \sigma dW_t + J dN(t).$$

- There is a form of Ito's lemmma for jump-diffusion processes. We will state it by considering a slightly more general process. Consider a process X_t, for $t \geq t_0 \equiv 0$, of the form:

$$X_t = X_{t_0} + \int_{t_0}^{t} b(u, X_{u-})du + \int_{t_0}^{t} \sigma(u, X_{u-})dW_u + \sum_{n=1}^{N(t)} \Delta X_n,$$

where $b(t, X_{t-})$ and $\sigma(t, X_{t-})$ are continuous nonanticipating processes with $\mathbb{E}_{t_0}^{\mathbb{Q}}[\int_{t_0}^{t} \sigma(u, X_{u-})^2 du] < \infty$, and where $\Delta X_n = X_{T_n} - X_{T_n -}$. Here, T_n, $n = 1, ..., N(t)$ denote the jump times of X_t. Note that the notation $\Delta X_n = X_{T_n} - X_{T_n -}$ reinforces a key point about the process X_t. It is right-continuous left-limits (RCLL or cadlag). This means that $X_t = \lim_{u \searrow t} X_u$ includes the effect of any jump at time t, whereas $X_{t-} = \lim_{u \nearrow t} X_u$ is the value just before a potential jump. Now consider the process $Y_t = f(t, X_t)$, where $f : [t_0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is any $C^{1,2}$ function.

- Then

$$\begin{aligned} f(t,X_t) &= f(t_0,X_{t_0}) + \int_{t_0}^{t} [\frac{\partial f}{\partial u}(u,X_{u-}) + \frac{\partial f}{\partial X}(u,X_{u-})b(u,X_{u-})]du \\ &+ \int_{t_0}^{t} \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(u,X_{u-})\sigma^2(u,X_{u-})du + \int_{t_0}^{t} \frac{\partial f}{\partial X}(u,X_{u-})\sigma(u,X_{u-})dW_u \\ &+ \sum_{n=1,T_n\le t}^{N(t)} (f(T_n,X_{T_n-}+\Delta X_n) - f(T_n,X_{T_n-})). \end{aligned}$$

In differential notation, this is written:

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t}(t,X_{t-})dt + \frac{\partial f}{\partial X}(t,X_{t-})b(t,X_{t-})dt \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(t,X_{t-})\sigma^2(t,X_{t-})dt + \frac{\partial f}{\partial X}(t,X_{t-})\sigma(t,X_{t-})dW_t \\ &+ (f(t,X_{t-}+\Delta X_t) - f(t,X_{t-}))dN(t), \end{aligned}$$

with $\Delta X_t = X_t - X_{t-}$.

- Proof (outline): Note that in the time intervals (T_n, T_{n+1}), the process $Y_t = f(t, X_t)$ is continuous and we can apply the standard form of Ito's lemma. If a jump of size ΔX_n occurs in X_t, then the resulting change in Y_t is simply $f(T_n, X_{T_n-} + \Delta X_n) - f(T_n, X_{T_n-})$. The total change can be written as the sum of the continuous and discontinuous parts.

- We have the process for log of the stock price $\log S(t)$:

$$\log S(t) = \log S(t_0) + (r - q - \frac{1}{2}\sigma^2 - k)t + \sigma W_t + \sum_{n=1}^{N(t)} J,$$

- What is the process for the stock price $S(t)$ itself?

- Here the function f is $f(t, \log S(t)) = \exp(\log S(t))$. We get

$$\begin{aligned} S(t) &= S(t_0) + \int_{t_0}^{t} (r-q-k) S(u-) du + \int_{t_0}^{t} \sigma S(u-) dW_u \\ &+ \sum_{n=1, T_n \le t}^{N(t)} (\exp(J) - 1) S(T_n -). \end{aligned}$$

Or in differential notation:

$$\frac{dS(t)}{S(t-)} = (r-q-k)dt + \sigma dW_t + (\exp(J) - 1) dN(t).$$

Note $k = \int_{-\infty}^{\infty} (e^x - 1)\nu(dx)$ which can be written as $k = \lambda \mathbb{E}^{\mathbb{Q}}[\exp(J) - 1]$, where this expectation should be interpreted as the expected value, under $\mathbb{Q}$, of $\exp(J) - 1$, conditional on a jump occuring and where λ is the intensity rate of $N(t)$ under $\mathbb{Q}$. So we can also write:

$$\frac{dS(t)}{S(t-)} = (r-q)dt + \sigma dW_t + (\exp(J) - 1) dN(t) - \lambda \mathbb{E}^{\mathbb{Q}}[\exp(J) - 1] dt.$$

- A slight digression:
- Suppose we consider the process Y_t, $t \geq t_0 \equiv 0$, with $Y_{t_0} \equiv 0$, defined via $Y_t \equiv \sum_{n=1}^{N(t)} f(J)$, for some well-behaved function f. Then what is $\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_t]$? We have:

$$\begin{aligned}
\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_t] &= \mathbb{E}_{t_0}^{\mathbb{Q}}[\sum_{n=1}^{N(t)} f(J)] = \mathbb{E}_{t_0}^{\mathbb{Q}}[\mathbb{E}_{t_0}^{\mathbb{Q}}[\sum_{n=1}^{N(t)} f(J)|N(t)=k]] = \mathbb{E}_{t_0}^{\mathbb{Q}}[\mathbb{E}_{t_0}^{\mathbb{Q}}[\sum_{n=1}^{k} f(J)]] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}}[k\mathbb{E}^{\mathbb{Q}}[f(J)]] = \lambda t \mathbb{E}^{\mathbb{Q}}[f(J)] = \lambda t \int_{-\infty}^{\infty} f(x) \frac{\nu(dx)}{\lambda} \\
&= t \int_{-\infty}^{\infty} f(x)\nu(dx).
\end{aligned}$$

Therefore, the process $\{Y_t - t\int_{-\infty}^{\infty} f(x)\nu(dx)\}$ i.e. the process $\{\sum_{n=1}^{N(t)} f(J) - t\int_{-\infty}^{\infty} f(x)\nu(dx)\}$ is a martingale.

- In differential notation, this is sometimes written:

$$\mathbb{E}^{\mathbb{Q}}_{t_0}[f(J)dN(t) - \lambda\mathbb{E}^{\mathbb{Q}}[f(J)]dt] = 0.$$

  Or, essentially equivalently, sometimes as:

$$\mathbb{E}^{\mathbb{Q}}_{t_0}[f(J)dN(t)] = \int_{-\infty}^{\infty} f(x)\nu(dx)dt.$$

- The choice $f(J) = \exp(J) - 1$ gives extra intuition on the final equation, two slides ago.

- There is a stochastic integral formula which resembles these last results:

- If we mean correct the Poisson process so it is a martingale, then stochastic integrals (which, financially, are the profit or loss from a self-financing trading strategy) with respect to the mean corrected Poisson process have zero expectation (which is completely analogous to the Brownian motion case).

- Is there an equivalent to the Black-Scholes PDE?
- Yes, but the derivation is different.
- Suppose we have an option that matures at time T at which time it pays $C(T)$. What is its price $C(t)$, at time t?
- We let $\hat{C}(t)$ be the forward price of the option, at time t to time T i.e. $\hat{C}(t) \equiv C(t)\exp(r(T-t))$. In the absence of arbitrage, the price of the option is its expected discounted value (under $\mathbb{Q}$). Therefore, $C(t) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-r(T-t))C(T)]$ which is the same as $\hat{C}(t) = \mathbb{E}_t^{\mathbb{Q}}[\hat{C}(T)]$, since $\hat{C}(T) = C(T)\exp(r(T-T)) = C(T)$. Therefore, $\hat{C}(t)$ is a martingale under $\mathbb{Q}$.
- Therefore the drift of $\hat{C}(t)$ is zero. But we can compute the drift by applying Ito's lemma. We apply it with $y(t) \equiv \log S(t)$ for convenience so $\hat{C}(t)$ (and also $C(t)$) is a function of $y(t)$ as well as of t.

- Ito gives us:

$$\begin{aligned} d\hat{C}(t) &= \frac{\partial \hat{C}}{\partial t}dt + (r - q - \frac{1}{2}\sigma^2 - k)\frac{\partial \hat{C}}{\partial y}dt + \frac{1}{2}\sigma^2\frac{\partial^2 \hat{C}}{\partial y^2}dt \\ &+ \sigma\frac{\partial \hat{C}}{\partial y}dW_t + (\hat{C}(t, y(t-) + \Delta y(t)) - \hat{C}(t, y(t-)))dN(t). \end{aligned}$$

Now take expectations under $\mathbb{Q}$. The expected value of the term $(\hat{C}(t, y(t-) + \Delta y(t)) - \hat{C}(t, y(t-)))dN(t)$ is simply $\int_{-\infty}^{\infty}(\hat{C}(t, y + x) - \hat{C}(t, y))\nu(dx)dt$. This follows from slide "Stochastic integral formula 2". Setting the expected value to zero, we get:

$$\begin{aligned} 0 &= \frac{\partial \hat{C}}{\partial t} + (r - q - \frac{1}{2}\sigma^2 - k)\frac{\partial \hat{C}}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 \hat{C}}{\partial y^2} \\ &+ \int_{-\infty}^{\infty}(\hat{C}(t, y + x) - \hat{C}(t, y))\nu(dx). \end{aligned}$$

- Now we switch back to $C(t)$. Since, $\hat{C}(t) \equiv C(t)\exp(r(T-t))$, then $\frac{\partial \hat{C}}{\partial t} = \frac{\partial C}{\partial t} - rC(t)$, and, furthermore, using $k = \int_{-\infty}^{\infty}(e^x - 1)\nu(dx)$ and rearranging, we get:

$$\begin{aligned} rC(t) &= \frac{\partial C}{\partial t} + (r - q - \frac{1}{2}\sigma^2)\frac{\partial C}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial y^2} \\ &+ \int_{-\infty}^{\infty}(C(t, y+x) - C(t,y) - (e^x - 1)\frac{\partial C}{\partial y})\nu(dx). \end{aligned}$$

  The first line is exactly the same as the Black-Scholes pde (remember we are in log co-ordinates: $y(t) \equiv \log S(t)$).

- The second line is new. With jumps, we get a non-local-term, an integral term. So we get a PIDE (partial-integro differential equation). In general, these are harder to solve than PDEs but they can be solved (very occasionally analytically but more typically numerically with some appropriate treatment of the integral term).

- For future reference, we switch back to stock price co-ordinates $S(t)$ and then we can write our pricing PIDE as:

$$\begin{aligned} rC(t,S(t)) &= \frac{\partial C}{\partial t}(t,S(t)) + (r-q)\frac{\partial C}{\partial S}(t,S(t))S + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2}(t,S(t))S^2 \\ &+ \int_{-\infty}^{\infty}(C(t,S(t)\exp(x)) - C(t,S(t)) - S(t)(e^x-1)\frac{\partial C}{\partial S})\nu(dx). \end{aligned}$$

or in terms of the forward stock price $\hat{S}(t)$, at time t to time T, defined via $\hat{S}(t) \equiv S(t)\exp((r-q)(T-t))$, and the forward option price $\hat{C}(t) \equiv \hat{C}(t,\hat{S}(t))$ as:

$$\begin{aligned} 0 &= \frac{\partial \hat{C}}{\partial t}(t,\hat{S}(t)) + (-k)\frac{\partial \hat{C}}{\partial \hat{S}}(t,\hat{S}(t))\hat{S} + \frac{1}{2}\sigma^2\frac{\partial^2 \hat{C}}{\partial \hat{S}^2}(t,\hat{S}(t))\hat{S}^2 \\ &+ \int_{-\infty}^{\infty}(\hat{C}(t,\hat{S}(t)\exp(x)) - \hat{C}(t,\hat{S}(t)))\nu(dx), \end{aligned}$$

where we used $k = \int_{-\infty}^{\infty}(e^x-1)\nu(dx)$ again.

- We have derived a counterpart to the Black-Scholes pde but we did not do so by a Merton-style delta-hedging argument. Nevertheless, it is instructive to see what happens if we try to follow a Merton-style delta-hedging argument. Specifically, we consider a self-financing trading strategy defined by holding one option (we will, again, work with the forward option price $\hat{C}(t)$) and selling ϕ_t units of stock in the forward market.

- One technical issue is that the process ϕ_t is a non-anticipating predictable (i.e. it is $\mathcal{F}_{t-}$ measurable) caglad process (i.e. it has right limits as opposed to the process for the stock price $S(t)$ (and for the forward stock price $\hat{S}(t)$) which has left limits).

- We hedge in the forward market, just for mathematical convenience, in order to avoid keeping track of terms involving r and q.
- We note that the dynamics of the forward stock price under $\mathbb{Q}$ are, by Ito's lemma:

$$\frac{d\hat{S}(t)}{\hat{S}(t-)} = (-k)dt + \sigma dW_t + (\exp(J) - 1)dN(t).$$

- We enter into the self-financing trading strategy at time t (at which time it costs nothing) and continue it until the option matures at time T, at all times holding one option (in the forward market) and being short ϕ_t units of stock (in the forward market).
- The value of the self-financing trading strategy at time T is:

$$\epsilon(\phi_t) \equiv \hat{C}(T) - \hat{C}(t) - \int_t^T \phi_u d\hat{S}(u).$$

- Using Ito's lemma, the value of the self-financing trading strategy at time T is:

$$
\begin{aligned}
\epsilon(\phi_t) &= \int_t^T [\frac{\partial \hat{C}}{\partial u}(u, \hat{S}(u-)) + \frac{\partial \hat{C}}{\partial \hat{S}}(u, \hat{S}(u-))(-k)\hat{S}(u-)]du \\
&+ \int_t^T \frac{1}{2}\frac{\partial^2 \hat{C}}{\partial \hat{S}^2}(u, \hat{S}(u-))\sigma^2 \hat{S}(u-)^2 du + \int_t^T \frac{\partial \hat{C}}{\partial \hat{S}}(u, \hat{S}(u-))\sigma \hat{S}(u-) dW_u \\
&+ \sum_{n=1, t<T_n \le T}^{N(T)} (\hat{C}(T_n, \hat{S}(T_n-) + \Delta \hat{S}(t)) - \hat{C}(T_n, \hat{S}(T_n-))) \\
&- \int_t^T \phi_u(-k)\hat{S}(u-)du - \int_t^T \phi_u \sigma \hat{S}(u-) dW_u \\
&- \sum_{n=1, t<T_n \le T}^{N(T)} \phi_{T_n}(\exp(J) - 1)\hat{S}(T_n-).
\end{aligned}
$$

- Now use the fact that $\hat{C}(t, \hat{S}(t))$ certainly satisfies our pricing PIDE and rearrange:

- The value of the self-financing trading strategy at time T is:

$$\begin{aligned}
\epsilon(\phi_t) &= \int_t^T (\frac{\partial \hat{C}}{\partial \hat{S}}(u, \hat{S}(u-)) - \phi_u)\sigma \hat{S}(u-)dW_u \\
&+ \sum_{n=1, t<T_n \leq T}^{N(T)} (\hat{C}(T_n, \hat{S}(T_n-) + \Delta \hat{S}(t)) - \hat{C}(T_n, \hat{S}(T_n-))) \\
&- \int_t^T \int_{-\infty}^{\infty} (\hat{C}(u, \hat{S}(u-)\exp(x)) - \hat{C}(u, \hat{S}(u-)))\nu(dx)du \\
&+ \int_t^T \phi_u k \hat{S}(u-)du - \sum_{n=1, t<T_n \leq T}^{N(T)} \phi_{T_n}(\exp(J) - 1)\hat{S}(T_n-).
\end{aligned}$$

Now the first line of this equation is a martingale and so are the second and third (using slide "Stochastic integral formula 2") taken together as, also, is the fourth line (this uses the result $k = \int_{-\infty}^{\infty}(e^x - 1)\nu(dx)$ again).

- So the expected value of the self-financing trading strategy under $\mathbb{Q}$ is zero i.e. $\mathbb{E}_t^{\mathbb{Q}}[\epsilon(\phi_t)] = 0$. This is actually true for any value of ϕ_t. But the key point about the Merton-style argument in the Black-Scholes (pure diffusion) framework is that not only is the expected value zero but, also with the choice $\phi_t = \frac{\partial \hat{C}}{\partial \hat{S}}$, the actual realised value is always zero i.e. the strategy is a perfect hedge.

- If we were to choose $\phi_t = \frac{\partial \hat{C}}{\partial \hat{S}}$, we would hedge out the risk from the Brownian component but not from the jump component. Therefore, it would **NOT** be a perfect hedge.

- What can we do?

- We can certainly compute the variance $\mathbb{V}_t^{\mathbb{Q}}[\epsilon(\phi_t)]$ of $\epsilon(\phi_t)$ under $\mathbb{Q}$. With the Merton-style argument in the Black-Scholes (pure diffusion) framework, the variance would be zero since it is a perfect hedge. What is it with jumps included?

- Using Ito's isometry formulae, and using the fact that $\mathbb{E}^{\mathbb{Q}}_t[\epsilon(\phi_t)] = 0$, we get:

$$\begin{aligned}\mathbb{V}^{\mathbb{Q}}_t[\epsilon(\phi_t)] &= \mathbb{E}^{\mathbb{Q}}_t[\int_t^T |(\frac{\partial \hat{C}}{\partial \hat{S}}(u, \hat{S}(u-)) - \phi_u)\sigma \hat{S}(u-)|^2 du] \\ &+ \mathbb{E}^{\mathbb{Q}}_t[\int_t^T \int_{-\infty}^{\infty} |\hat{C}(u, \hat{S}(u-)\exp(x)) - \hat{C}(u, \hat{S}(u-)) \\ &- \phi_u(\exp(x) - 1)\hat{S}(u-)|^2 \nu(dx)du].\end{aligned}$$

- Note this is a positive quadratic function of ϕ_t.

- It is instrutive to ask what value of ϕ_t minimises the variance $\mathbb{V}^{\mathbb{Q}}_t[\epsilon(\phi_t)]$. We can find this out by differentiating $\mathbb{V}^{\mathbb{Q}}_t[\epsilon(\phi_t)]$ with respect to ϕ_t and setting the resulting equation to zero.

- We get (after trivial algebraic rearrangement):

$$
\begin{aligned}
0 &= (\frac{\partial \hat{C}}{\partial \hat{S}}(t, \hat{S}(t-)) - \phi_t)\sigma^2 \hat{S}^2(t-) + (\int_{-\infty}^{\infty} (\hat{C}(t, \hat{S}(t-)\exp(x)) - \hat{C}(t, \hat{S}(t-)) \\
&- \phi_t(\exp(x) - 1)\hat{S}(t-))(\exp(x) - 1)\hat{S}(t-)\nu(dx)).
\end{aligned}
$$

  Solving for ϕ_t, we get:

$$
\phi_t = \frac{\sigma^2 \frac{\partial \hat{C}}{\partial \hat{S}}(t, \hat{S}(t-)) + \frac{1}{\hat{S}(t-)}(\int_{-\infty}^{\infty}(\hat{C}(t, \hat{S}(t-)\exp(x)) - \hat{C}(t, \hat{S}(t-)))(\exp(x) - 1)\nu(dx))}{\sigma^2 + \int_{-\infty}^{\infty}(\exp(x) - 1)^2 \nu(dx)}.
$$

- Note this is the optimal hedge and it is valid for all t up to the option maturity T - not just at the time the self-financing trading strategy is entered into or the option is first bought - although, clearly, ϕ_t will change as the stock price changes i.e. it is a dynamic strategy.

- Note that for a forward contract: $\hat{C}(t,\hat{S}(t-)) = \hat{S}(t-)$ (ignoring the fixed leg), and so $(\hat{C}(t,\hat{S}(t-)\exp(x)) - \hat{C}(t,\hat{S}(t-)))(\exp(x)-1) = \hat{S}(t-)(\exp(x)-1)^2$. Furthermore, $\frac{\partial \hat{C}}{\partial \hat{S}}(t,\hat{S}(t-)) = 1$ and so the optimal hedge is $\phi_t \equiv 1$ for all t. It is clear that, for this special case, that $\mathbb{V}_t^{\mathbb{Q}}[\epsilon(\phi_t)] = 0$. This is just the same buy-and-hold strategy as one uses in the pure diffusion case. So we can hedge forward contracts i.e. linear contracts perfectly but non-linear contracts (eg. options) are a different matter.

- It is clear that it is not possible to hedge non-linear contracts perfectly for the jump-diffusion process we have considered.

- It is instructive to see if there are any special cases of our jump-diffusion process for which perfect hedging of non-linear contracts is possible. It turns out that there are only two such special cases:

- The first special case is when there are no jumps i.e. $\nu(dx) = 0$. Then the optimal hedge ϕ_t becomes $\phi_t = \frac{\partial \hat{C}}{\partial \hat{S}}$. This the just the standard delta hedge. Clearly, in this special case, $\mathbb{V}_t^{\mathbb{Q}}[\epsilon(\phi_t)] = 0$.

- The second special case is when $\sigma = 0$ and when it is only possible to have jumps of a constant size a, say. Then $\nu(dx) = \lambda\delta(a)$, where δ denotes the Dirac delta function. Then the the optimal hedge ϕ_t becomes:

$$\phi_t = \frac{\hat{C}(t, \hat{S}(t-)\exp(a)) - \hat{C}(t, \hat{S}(t-))}{(\exp(a) - 1)\hat{S}(t-)}.$$

  Again, it is easy to verify that, in this special case, $\mathbb{V}_t^{\mathbb{Q}}[\epsilon(\phi_t)] = 0$. Note this model is clearly financially unrealistic since the stock price can only move ($\sigma = 0$, so it has no Brownian component) by a fixed amount (ignoring the deterministic drift).

- Apart from these two special cases, perfect hedging of non-linear contracts is not possible.
- A market is called complete if it possible to perfectly hedge all derivatives. Since this is not possible (except in the two special cases of our jump-diffusion process already mentioned), we can conclude that our market is incomplete.
- Note that it may be theoretically possible to form a complete market, in some circumstances, by extending the set of possible hedging instruments to include (a possibly infinite set of) other derivatives such as vanilla options (and not just the stock price (or the forward stock price)).
- A derivative is called "redundant" if it can be replicated (which is just the same as hedging with a minus sign) by a self-financing trading strategy in the underlying stock. Therefore, in a complete market, all derivatives are redundant.

- With jump-diffusion models, in general (i.e. except in the two special cases of our jump-diffusion process), derivatives are not redundant. This means that if, for example, we have written an exotic option then simpler options (such as vanilla options) have a role to play in trying to reduce the hedging error associated with the exotic option.

- Monte Carlo simulation studies show that the hedging error can, often, be significantly reduced by taking a static (i.e. a buy-and-hold) position in even a small number of vanilla options.

- The hedging error may be further reduced by allowing dynamic positions in the vanilla options (although, in practice, since options have much wider bid-offer spreads than the underlying stock, this may result in large transactions costs).

- The most well-known and, in fact, the first jump-diffusion model is the Merton (1976) model. We will examine a form of this model in greater depth later.

- It is the model above with the choice: $\frac{\nu(dx)}{\lambda} = \frac{1}{\Sigma\sqrt{2\pi}} \exp(-\frac{1}{2}\frac{(x-\overline{\mu})^2}{\Sigma^2})$, where $\overline{\mu}$ and Σ are constants (with $\Sigma \geq 0$).

- Since this is the density of a Gaussian distribution, we see that the jump distribution is Gaussian which, in turn implies that X_t, and hence also $\log S(t)$, is, conditional on the number of jumps $N(t)$ until time t, normally distributed.

- Unsuprisingly, this results in a very tractable model for which Monte Carlo simulation is particularly straightforward.

- If the parameter $\overline{\mu}$ (which is the mean jump size) is negative, it results in the distribution for X_t being negatively skewed (which is typically what is observed in the equity options markets).

- The defining feature of a compound Poisson process is that there are a finite number of jumps in any finite time interval. We recognised this when we said we required λ (which is the expected number of jumps per unit of time) to satisfy $0 < \lambda < \infty$. Recall that we wrote that we draw an independent and identically distributed random variable J (which, as a special case, could be a constant, not necessarily one - nor necessarily positive) from a given distribution for which the probability of J being in the (Borel) set A is $\text{Prob}(J \in A) = \nu(A)/\lambda$, further implying $\nu(\mathbb{R}) = \lambda$.

- This means that the measure $\nu(dx)$ is not a true probability distribution (it integrates to λ - and not one) but it does integrate to one (and, therefore, is a true probability distribution) when we normalise to one by dividing by λ (which we can do since λ is finite).

- This suggests that if we wish to go beyond compound Poisson processes, we need to consider processes for which there are an infinite number of jumps in any finite time interval.

- Definition:

- A Lévy process is a process X_t, $t \geq t_0 \equiv 0$, with $X_{t_0} = 0$, with the following three properties:
  1./ Stationary increments:
  The distribution of $X_{t+s} - X_t$, for any $t \geq 0$, depends only on s and not upon t, for all $s \geq 0$.
  2./ Independent increments:
  For every possible set of times $0 \leq t_1 < t_2 \leq t_3 < t_4$, $X_{t_4} - X_{t_3}$ is independent of $X_{t_2} - X_{t_1}$.
  3./ The following technical condition is satisfied: For all $\epsilon > 0$,

$$\lim_{h \to 0} \text{Prob}(|X_{t+h} - X_t| \geq \epsilon) = 0.$$

- The third condition basically says that jumps happen at random times (and that for any given time t, the probability of a jump occuring at time t is zero) and it rules out jumps at fixed or non-random times.

Infinite activity Lévy processes

- Actually, compound Poisson processes (and jump-diffusion processes) satisfy all three of these conditions so they are also examples of Lévy processes. However, in general, Lévy processes do not need to have a finite number of jumps in any finite time interval as is required for compound Poisson processes (and jump-diffusion processes).

- If they have an infinite number of jumps in any finite time interval, then they are called "infinite activity" Lévy processes.

- We need a way of counting the number of jumps.

- Definition:

- Let X_t be a Lévy process. The Lévy measure ν is defined by:

$$\nu(A) = \mathbb{E}[\#t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A].$$

  In words, $\nu(A)$ is the expected number of jumps, per unit of time, whose size belongs to the set A.

- The measure ν is not necessarily a finite measure but it is always finite on any compact set A which does not include zero.

- In other words, a Lévy process can have an infinite number of infinitesimally small jumps but only a finite number of jumps of finite size (else it would not satisfy condition 3./).

- This leads to the Lévy-Ito decomposition:

- Let X_t, $t \geq t_0 \equiv 0$, with $X_{t_0} = 0$, be a Lévy process and ν its Lévy measure:
- Then ν satisfies $\int_{-\infty}^{\infty} \min\{1, x^2\}\nu(dx) < \infty$. Furthermore:
- There exists γ and a Brownian motion W_t, with $W_{t_0} = 0$, with volatility σ, say, such that

$$X_t = \gamma t + \sigma W_t + X_t^{\ell} + \lim_{\epsilon \searrow 0} \tilde{X}_t^{\epsilon},$$

  where X_t^{ℓ} is a compound Poisson process $\sum_{n=1,|X_s|\geq 1}^{N(t)} \Delta X_s$, with $X_{t_0}^{\ell} = 0$, (i.e. with jumps of magnitude greater than or equal to one) and where $\tilde{X}_t^{\epsilon}$ can be seen to be an infinite sum of compensated (i.e. mean corrected so they are martingales) compound Poisson processes with jumps whose magnitude is less than one and greater than or equal to ϵ.
- Furthermore, X_t^{ℓ} and $\tilde{X}_t^{\epsilon}$ are independent (as is W_t).
- The intuition here is as follows: We have already shown that the sum of independent Poisson processes is also a Poisson process. This suggests the idea of building blocks.

- The intuition is:

- There are a finite number of jumps of magnitude greater than or equal to one (the threshold does not have to be one - it could be any threshold strictly greater than zero). Represent them as a compound Poisson process. There may be (and will be, if the Lévy process has infinite activity) an infinite number of jumps of magnitude less than one. Their sum may not converge. But if we compensate them (i.e. mean correct them), it turns out that the sum does converge.

- The term involving γ is, essentially, a "drift" term.

- Mathematically, we say that a Lévy process has infinite activity if:

$$\int_{-1}^{1} \nu(dx) = \infty.$$

- I have already defined what an infinitely divisible distribution is:

- A probability distribution, with characteristic function $\phi(z)$ is said to be infinitely divisible if, for every integer $n \geq 1$, $\phi(z)$ is also the n^{th} power of a characteristic function.

- It is not difficult to see that all Lévy process must have an infinitely divisible distribution. This is because, for any integer $n \geq 1$,

$$X_t = X_{t/n} + [X_{2t/n} - X_{t/n}] + \ldots + [X_t - X_{(n-1)t/n}],$$

and so, intutively, it is clear that:

$$\begin{aligned} \mathbb{E}_{t_0}[\exp(iz(X_t))] &= \mathbb{E}_{t_0}[\exp(iz(X_{t/n} + [X_{2t/n} - X_{t/n}] + \ldots + [X_t - X_{(n-1)t/n}]))] \\ &= (\mathbb{E}_{t_0}[\exp(iz(t/n)(X_{t/n}))])^n = (\exp((t/n)\Phi(z)))^n, \end{aligned}$$

by stationary and independent increments, for some function $\Phi(z)$, independent of t.

- Setting $n = 1$, implies that the characteristic function of X_t must be of the form:

$$\mathbb{E}_{t_0}[\exp(iz(X_t))] = \exp(t\Phi(z)).$$

  We see that $\Phi(z)$ is the "characteristic exponent" that I defined earlier.

- Now all we need is the form of the characteristic exponent.

- I previously wrote down (on slide "Jump-diffusion processes 5") the form of the characteristic exponent for a jump-diffusion process, namely:

$$iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1)\nu(dx).$$

- I also wrote down (on slide "Jump-diffusion processes 6") the characteristic exponent of the process $M_t = X_t - \lambda m_J t$, where m_J is the expectation of J, namely:

$$iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx)\nu(dx).$$

- Why?

- The Lévy-Ito decomposition says we can represent a Lévy process as drift plus Brownian motion plus an independent compound Poisson process with jumps of magnitude greater than or equal to one plus an infinite sum of compensated (i.e. mean corrected) independent compound Poisson processes with jumps of magnitude less than one.

- The processes are all independent so characteristic functions multiply. Therefore, characteristic exponents add.

- Therefore, the characteristic exponent $\Phi(z)$ of any Lévy process X_t must be of the form:

$$\Phi(z) \;=\; iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty}(e^{izx}-1)\mathbf{1}_{|x|\geq 1}\nu(dx) + \int_{-\infty}^{\infty}(e^{izx}-1-izx)\mathbf{1}_{|x|<1}\nu(dx),$$

where $\mathbf{1}$ denotes the indicator function. This can also be expressed as:

$$\Phi(z) = iz\gamma - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty}(e^{izx}-1-izx\mathbf{1}_{|x|<1})\nu(dx).$$

This formula is called the "Lévy-Khinchin" representation.

- It tells us, given the Lévy measure, how to compute the characteristic exponent of the Lévy process.

- There are several Lévy processes (eg. Variance Gamma, CGMY (named after Carr, Geman, Madan and Yor), Normal Inverse Gaussian) which have become used in mathematical finance. The only one we will have time to look at is the CGMY process. The CGMY process has a Lévy measure defined by:

$$\nu(dx) = \frac{C\exp(-M|x|)}{|x|^{1+Y}}\ , \text{ for } x > 0, \ \ \nu(dx) = \frac{C\exp(-G|x|)}{|x|^{1+Y}}\ , \text{ for } x < 0,$$

  where $C > 0$, $M > 0$, $G > 0$ and $Y < 2$ are constants (the condition $Y < 2$ is required to ensure ν satisfies $\int_{-\infty}^{\infty} \min\{1, x^2\}\nu(dx) < \infty$).

- We can see that if $M = G$, then the Lévy measure is symmetric. So the difference between M and G controls the asymmetry or skewness of the CGMY process.

- One can easily show (by checking whether $\int_{-1}^{1} \nu(dx) = \infty$) that if $Y \geq 0$, then the CGMY process has infinite activity. If $Y < 0$, then the process has a finite number of jumps in a finite time interval and is, therefore, a compound Poisson process.

- We can compute the characteristic exponent of the CGMY process by using the Lévy-Khinchin representation (it follows from some standard results involving the gamma function - the gamma function is a special function in mathematics - a series representation allows it to be rapidly computed - it generalises the factorial function to reals - for an integer n, $\Gamma(1+n) = n!$). Ignoring, for a moment, the drift and Brownian components, the characteristic exponent is:

$$C\Gamma(-Y)((M-iz)^Y - M^Y + (G+iz)^Y - G^Y).$$

- Note how simple it is (strictly speaking, the above formula only applies if $Y \neq 0$ and $Y \neq 1$ - slightly different forms apply for these special cases although the characteristic exponent is always well-behaved).

- The CGMY process has an interesting property. In the special case that we set $M = 0$, $G = 0$ and we let $Y \nearrow 2$, the characteristic exponent tends to $4C\Gamma(-2)(-\frac{1}{2}z^2)$, which is the characteristic exponent of Brownian motion. So CGMY includes Brownian motion as a limiting case.

- One can calibrate a CGMY process to the market prices of vanilla options.

- If you do this, you find that one can typically set the volatility σ of the Brownian motion component to be zero and get just as good a calibration (or, more specifically, one gets just as good a calibration in the case of infinite activity).

- This is not mathematically necessary, mathematically one can have $\sigma > 0$ (regardless of the value of Y, $Y < 2$), - its an empirical observation.

- The intuition here is that the CGMY process is a very rich process. It can capture occasional large jumps. Additionally, if $Y \geq 0$ (so it has infinite activity), it can capture the idea of infinitesimally small moves happening with infinite frequency - which, intuitively, is similar to the effect of Brownian motion. Therefore, the separate Brownian motion component typically becomes redundant.

- If we want to model the dynamics of stock prices under $\mathbb{Q}$, then as with the jump-diffusion process, we model log prices.

- We assume that the stock price, $S(t)$, at time t, with $t \geq t_0 \equiv 0$, $X_{t_0} = 0$, is:

$$S(t) \;=\; S(t_0)\exp((r-q)t)\exp(X_t).$$

  This is of the same general form as for the jump-diffusion process.

- Again, we must specify γ so that the drift rate on the stock, under $\mathbb{Q}$, is equal to $r - q$, or equivalently, so that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$. This requires the choice:

$$\gamma = -\frac{1}{2}\sigma^2 - \int_{-\infty}^{\infty}(e^x - 1 - x\mathbf{1}_{|x|<1})\nu(dx),$$

  because then setting $iz = 1$ in the characteristic function shows that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$.

- When we use this value of γ, we call it the mean corrected characteristic function.
- It can be shown that all Lévy processes (except Browian motion with no jumps) generate excess kurtosis which in turn implies they produce curvature in the implied volatility surface.
- They can produce (depending on the parameters of the process) skew in the implied volatility surface (for the CGMY model, if $M > G$, we get negative skewness which is typically what is observed in the equity options markets).
- In order to calibrate a Lévy process to market option prices, we need a fast way to price vanilla options.
- Fortunately, such a way exists.

- The value of an option, at time $t_0 \equiv 0$, when the asset price is $S(t_0)$, whose payoff function is $\min\left(S(T), K\right)$, where $S(T)$ is the asset price at maturity T and K is the strike, is given by

$$
\begin{aligned}
f(S(t_0), K, T) &= \frac{1}{\pi}\sqrt{S(t_0)K}e^{-\frac{T}{2}(r+q)}\int_0^{+\infty}\mathrm{Re}\left(e^{iuk}\frac{\phi_T(-u-i/2)}{u^2+1/4}\right)du,\\
&= \frac{1}{\pi}\sqrt{S(t_0)K}e^{-\frac{T}{2}(r+q)}\int_0^{\pi}\mathrm{Re}\left(e^{i\tan(y/2)k/2}\Phi_T\left(-\frac{1}{2}\tan\left(-\frac{y}{2}\right)-\frac{i}{2}\right)\right)\,dy,
\end{aligned}
$$

  where $k = \log(K/S(t_0)) - (r-q)T$, ϕ_T is the mean corrected characteristic function of the Lévy process, and where in the second line we have made the substitution $y = 2\arctan(2u)$.

- Call and put vanilla option prices, at time $t_0 \equiv 0$, are given by $S(t_0)e^{-qT} - f(S(t_0), K, T)$ and $Ke^{-rT} - f(S(t_0), K, T)$ respectively.

- The only reason for making the substitution $y = 2\arctan(2u)$ is that it can sometimes be convenient, when numerically integrating, to avoid having an infinite limit.

- This formula can easily be implemented and can be used to price, say, 30 vanilla options in less than around 50 milliseconds.

- I displayed several forms of the pricing PIDE for the case of a jump-diffusion process about 30 slides ago.

- It turns out that the forms which explicitly use the term k do not necessarily work for infinite activity Lévy processes (this is a rather technical issue to do with the convergence of the infinite number of jumps). However, the forms which do not explicitly use the term k, **do** apply for all Lévy processes (including all cases of infinite activity).

- The best form for numerical implementation is:

$$
\begin{aligned}
rC(t) &= \frac{\partial C}{\partial t} + (r - q - \frac{1}{2}\sigma^2)\frac{\partial C}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial y^2} \\
&+ \int_{-\infty}^{\infty} (C(t, y + x) - C(t, y) - (e^x - 1)\frac{\partial C}{\partial y})\nu(dx),
\end{aligned}
$$

  where $y(t) \equiv \log S(t)$.

- This pricing PIDE can be solved numerically to price exotic options.

- There is a form of Girsanov's Theorem for Lévy processes.
- The only observation I make is this:
- In pure diffusion models, only the drift can change - the Brownian motion volatility σ never changes.
- With processes with jumps, the Brownian motion volatility σ also never changes. However, in general (there is the occasional exception) all the other parameters of a Lévy process can change under a change of measure - not just the drift.
- In fact, a process which is a Lévy process in, say, a risk-neutral equivalent martingale measure may not even be a Lévy process in the real-world physical measure.

- Given that Lévy processes must have stationary and independent increments, a natural question is what happens if we relax one of these requirements or more generally, how can we move beyond Lévy processes and what might we achieve by doing so.

- A key observation is, because of the fact that the characteristic function of a Lévy process is always of the form $\exp(t\Phi(z))$, where $\Phi(z)$ is independent of t, there are always parameters which scale naturally with time.

- For Brownian motion, this parameter is σ^2, for compound Poisson processes, it is λ, for jump-diffusion processes, it is both σ^2 and λ, for the CGMY process, it is C (and σ^2 if non-zero).

- One way to go beyond Lévy processes, is to introduce the idea of a stochastic time-change. This can be thought of as a generalisation of stochastic variance (or stochastic volatility, as it is usually called). This is a little more advanced than we can cover today (although, we will briefly look at Cox processses in connection with jumps to default and credit risk which are essentially the same idea).

- A simpler way to move beyond Lévy processes is to have what is, essentially, a deterministic time-change. This relaxes the requirement of stationary increments but keeps the requirement of independent increments. This is essentially achieved by making the parameters, which scale naturally with time, time-dependent i.e. make these parameters deterministic functions of time t.

- We will do this for a jump-diffusion process with volatility $\sigma(t)$ and intensity rate $\lambda(t)$ at time t, with $0 < \sigma(t) < \infty$ and $0 < \lambda(t) < \infty$, for all $t \geq t_0 \equiv 0$.

- A rough rule-of-thumb is that if we do this and we are interested in the distribution of the process at a given time t (for example, pricing a vanilla (standard European) option), then if we replace all previous references to $\sigma^2 t$ and λt by $\int_{t_0}^{t} \sigma^2(s)ds$ and $\int_{t_0}^{t} \lambda(s)ds$ then our previous results are still valid (likewise we can introduce deterministic term structures of interest-rates and dividend yields).

- Note this is definitely **NOT** true if pricing path-dependent options eg. barrier options.

- Consider a time-inhomogeneous jump-diffusion process. We assume that, under $\mathbb{Q}$, the stock price, $S(t)$, at time t, with $t \geq t_0 \equiv 0$, $X_{t_0} = 0$, evolves as:

$$
\begin{aligned}
S(t) &= S(t_0) \exp\left(\int_{t_0}^{t} (r(s) - q(s))ds\right) \exp(X_t) \\
&= S(t_0) \exp\left(\int_{t_0}^{t} (r(s) - q(s))ds\right) \exp\left(\int_{t_0}^{t} \gamma(s)ds + \int_{t_0}^{t} \sigma(s) W_s + \sum_{n=1}^{N(t)} J\right),
\end{aligned}
$$

where $r(t)$ is the instantaneous risk-free interest-rate and $q(t)$ is the dividend yield at time t. Furthermore, $N(t)$, $N(t_0) = 0$, is a time-inhomogeneous Poisson process with intensity rate $\lambda(t)$ at time t. It has independent (but not stationary) increments. The probability that $N(t) = n$ is given by:

$$
\text{Prob}(N(t) = n) = \frac{\exp(-\int_{t_0}^{t} \lambda(s)ds)(\int_{t_0}^{t} \lambda(s)ds)^n}{n!}.
$$

- Note that X_t has independent increments (condition 2. of our definition of a Lévy process) but it does not have stationary increments (condition 1. of our definition of a Lévy process). We will refer to the process for X_t as a time-inhomogeneous jump-diffusion process.

- We need to specify the distribution of jumps J. We choose a special case of the Merton (1976) model in which only one jump size is allowed and that size is $-\infty$. By Ito's lemma, we have:

$$\frac{dS(t)}{S(t-)} = (r(t) - q(t) + \gamma(t))dt + \sigma(t)dW_t + (\exp(-\infty) - 1)dN(t).$$

  Or:

$$dS(t) = (r(t) - q(t) + \gamma(t))S(t-)dt + \sigma(t)S(t-)dW_t - S(t-)dN(t).$$

- We see that when a jump occurs in $N(t)$, the stock prices goes to zero and stays there forever.

- The probability, under $\mathbb{Q}$, that this does not occur by time t is the probability that there are zero jumps in $N(t)$ between time $t_0 \equiv 0$ and time t, which is:

$$\mathrm{Prob}(N(t)=0) = \frac{\exp(-\int_{t_0}^{t}\lambda(s)ds)(\int_{t_0}^{t}\lambda(s)ds)^0}{0!} = \exp(-\int_{t_0}^{t}\lambda(s)ds).$$

Therefore, the probability that it does occur by time t is:

$$\mathrm{Prob}(N(t)>0) = 1-\exp(-\int_{t_0}^{t}\lambda(s)ds).$$

We can compute the characteristic function of X_t. It is:

$$\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iz(X_t))] \;=\; \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iz(\int_{t_0}^{t}\gamma(s)ds+\int_{t_0}^{t}\sigma(s)dW_s+\sum_{n=1}^{N(t)}(-\infty)))].$$

- $$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(X_t))] &= \exp(iz\int_{t_0}^{t}\gamma(s)ds - \frac{1}{2}z^2\int_{t_0}^{t}\sigma(s)^2 ds) \\
&\quad \Big\{(\exp(-\int_{t_0}^{t}\lambda(s)ds))\mathbb{E}^{\mathbb{Q}}_{t_0}[\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(\sum_{k=1}^{N(t)}(-\infty)))|N(t)=0]] \\
&+ (1-\exp(-\int_{t_0}^{t}\lambda(s)ds))\mathbb{E}^{\mathbb{Q}}_{t_0}[\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(\sum_{k=1}^{N(t)}(-\infty)))|N(t)>0]]\Big\} \\
&= \exp(iz\int_{t_0}^{t}\gamma(s)ds - \frac{1}{2}z^2\int_{t_0}^{t}\sigma(s)^2 ds) \\
&\quad \Big\{(\exp(-\int_{t_0}^{t}\lambda(s)ds))\exp(iz0) + (1-\exp(-\int_{t_0}^{t}\lambda(s)ds))\exp(iz(-\infty))\Big\}.
\end{aligned}$$

- Or:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(X_t))] &= \exp(iz\int_{t_0}^{t}\gamma(s)ds - \frac{1}{2}z^2\int_{t_0}^{t}\sigma(s)^2 ds)\\ &\quad \big\{(\exp(-\int_{t_0}^{t}\lambda(s)ds)) + (1-\exp(-\int_{t_0}^{t}\lambda(s)ds))\exp(iz(-\infty))\big\}.\end{aligned}$$

(We cannot simplify further the term $\exp(iz(-\infty))$ as it is indeterminate if z is purely real).

- We must choose $\gamma(t)$ so that the stock price $S(t)$ has drift $r(t)-q(t)$ under $\mathbb{Q}$ i.e. such that $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(X_t))] = 1$.

- It is easy to see, from the form of the characteristic function (with $z=-i$), that we must choose:

$$\gamma(t) = -\frac{1}{2}\sigma(t)^2 + \lambda(t)$$

- With this choice, we can see the dynamics of the stock price $S(t)$ under $\mathbb{Q}$ are:

$$dS(t) = (r(t) - q(t) + \lambda(t))S(t-)dt + \sigma(t)S(t-)dW_t - S(t-)dN(t).$$

Or:

$$S(t) = S(t_0)\exp\Big(\int_{t_0}^{t}(r(s) - q(s) - \frac{1}{2}\sigma(s)^2 + \lambda(s))ds\Big)\exp\Big(\int_{t_0}^{t}\sigma(s)W_s + \sum_{n=1}^{N(t)}(-\infty)\Big).$$

- Note that we could also have established the same equations by using the equivalent formulae we derived for the jump-diffusion process earlier and substituting the measure $\nu(dx) = \lambda(t)\delta(-\infty)$, where δ denotes the Dirac delta function.

- Observing the form of the last two equations, we see that until the time of the first jump in $N(t)$, the stock price dynamics resemble those of an asset whose dividend yield is effectively $q(t) - \lambda(t)$. Therefore, if we wish to price a vanilla call option, we can simply use the Black-Scholes formula with the same adjustment. The price $C(t_0, S(t_0), t)$, at time $t_0 \equiv 0$, of a vanilla call option which pays off $\max(S(t) - K, 0)$ at its maturity t is simply:

$$
\begin{aligned}
& C(t_0, S(t_0), t) = \mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(-\int_{t_0}^{t} r(s)ds) \max(S(t) - K, 0)] \\
= \; & \exp(-\int_{t_0}^{t} \lambda(s)ds)\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(-\int_{t_0}^{t} r(s)ds) \max(S(t) - K, 0)|N(t) = 0] \\
+ \; & (1 - \exp(-\int_{t_0}^{t} \lambda(s)ds))\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(-\int_{t_0}^{t} r(s)ds) \max(S(t) - K, 0)|N(t) > 0] \\
= \; & \exp(-\int_{t_0}^{t} \lambda(s)ds)\big\{S(t_0) \exp(-\int_{t_0}^{t} (q(s) - \lambda(s))ds)N(d_1) - K \exp(-\int_{t_0}^{t} r(s)ds)N(d_2)\big\}.
\end{aligned}
$$

- This can also be expressed as:

$$C(t_0, S(t_0), t) = S(t_0)\exp(-\int_{t_0}^{t} q(s)ds)N(d_1) - K\exp(-\int_{t_0}^{t}(r(s)+\lambda(s))ds)N(d_2),$$

where:

$$\begin{aligned} d_1 &= \frac{\log(S(t_0)/K) + \int_{t_0}^{t}(r(s) - q(s) + \lambda(s) - \frac{1}{2}\sigma^2(s))ds + \int_{t_0}^{t}\sigma^2(s)ds}{\sqrt{\int_{t_0}^{t}\sigma^2(s)ds}}, \\ d_2 &= d_1 - \sqrt{\int_{t_0}^{t}\sigma^2(s)ds}. \end{aligned}$$

- We have written down lots of equations but let us pause for intuition:

- We have a stock price that follows geometric Brownian motion until the time of the first jump in $N(t)$ and afterwards it is identically equal to zero.

- This is essentially a model of default.

- We can say that, until the time of the first jump in $N(t)$, the company, whose stock price we are modelling, is solvent. At the instant that the first jump occurs in $N(t)$, the company becomes bankrupt and its shares are worthless.

- In other words, we model the time to default as the first time there is a jump in the process $N(t)$.

- We can show an interesting result by examining the formula for the price $C(t_0, S(t_0), t)$ of a vanilla call option: It looks like the Black-Scholes formula with, essentially, an interest rate of $r(t) + \lambda(t)$. We know $\lambda(t) > 0$, for all t, and we know that the price of a call option increases with increasing interest-rates, so we can say that the price of a vanilla call option on a stock that is defaultable is worth more than the price of a vanilla call option on a stock that is assumed to be non-defaultable, all other things being equal.

- This result was first proven in Merton (1976).

- We can obtain the price of a put option with the same strike and maurity by put-call parity: It is $C(t_0, S(t_0), t) + K\exp(-\int_{t_0}^{t} r(s)ds) - S(t_0)\exp(-\int_{t_0}^{t} q(s)ds)$

- Note that the prices of both call and put options do assume, that while the company underlying the stock can default, the option writer cannot (for example, it might be an exchange-traded option on an exchange with very good credit quality and, perhaps, some collateral arrangements in force).

- We have said that we can model the time to default as the first time there is a jump in the process $N(t)$.

- Let τ be the (random) time to default, with $\tau > t_0 \equiv 0$. Then:

$$\text{Prob}(\tau > t) = \text{Prob}(N(t) = 0) = \exp(-\int_{t_0}^{t} \lambda(s)ds),$$

  is simply the probability (under $\mathbb{Q}$) that the stock does not default before time t.

- Suppose the company underlying the stock has issued a zero-coupon risky bond which promises to pay one at maturity t. However, suppose, in the event of default, at or before time t, it only pays δ, which is a constant with $\delta < 1$, at maturity t. In the absence of arbitrage, the price $D(t_0, t, \delta)$, at time t_0, of the risky bond must be:

$$\begin{aligned} D(t_0, t, \delta) &= \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-\int_{t_0}^{t} r(s)ds) \{\text{Prob}(\tau > t)1 + (1 - \text{Prob}(\tau > t))\delta\}] \\ &= \exp(-\int_{t_0}^{t} r(s)ds) \left\{\delta + (1-\delta)\exp(-\int_{t_0}^{t} \lambda(s)ds)\right\}. \end{aligned}$$

- Alternatively, if I could observe the price $D(t_0, t, \delta)$ in the market, I could solve for $\exp(-\int_{t_0}^{t} \lambda(s)ds)$ via:

$$\int_{t_0}^{t} \lambda(s)ds = \log\left(\frac{1-\delta}{(D(t_0,t,\delta)/\exp(-\int_{t_0}^{t} r(s)ds)) - \delta}\right).$$

- Furthermore, if I could observe the prices in the market of a number of such zero coupon risky bonds with different maturities, and if I assumed some functional form for $\lambda(t)$ (such as piecewise constant in the time intervals defined by successive risky bond maturities), then I could recursively solve for the functional form for $\lambda(t)$, for all t, by using the above formula sucessively for increasing t.

- Clearly, I need to know $\exp(-\int_{t_0}^{t} r(s)ds)$ (which I can get as they are the prices of risk-free zero coupon bonds) and I need to have an estimate of δ, the recovery in default (which I may be able to get from historical data).

- Having solved for $\lambda(t)$, for all t, I could also solve for $\sigma(t)$ (again, assumed piecewise constant), for all t, by matching the market prices of vanilla stock options of different maturities and a given strike (say, at-the-money-forward).

- We know that the model will generate an implied volatility which is negatively skewed (although it may or may not accord closely with that observed in the market).

- We stress one feature of this model:

- Unlike, when we were modelling jump-diffusion processes and we needed to consider all the jumps of $N(t)$, in this default model, we are only interested in modelling the first jump of $N(t)$. At the time of the first jump of $N(t)$, the stock price goes to zero and stays there and, essentially, our pricing problem has finished.

- One further point: The price $D(t_0, t, \delta)$, at time t_0, of the zero coupon risky bond is:

$$
\begin{aligned}
D(t_0, t, \delta) &= \exp(-\int_{t_0}^{t} r(s)ds) \left\{ \delta + (1-\delta) \exp(-\int_{t_0}^{t} \lambda(s)ds) \right\} \\
&= \exp(-\int_{t_0}^{t} r(s)ds) \left\{ \delta + (1-\delta) \mathrm{Prob}(\tau > t) \right\}.
\end{aligned}
$$

In words, the payoff of the zero coupon risky bond is equivalent to receiving a risk-free payment of δ and an additional payment of $1-\delta$ conditional on no default occuring before time t. All payments are made at the bond maturity t. This assumption on the recovery is called the recovery-of-treasury assumption (other assumptions are possible).

- Suppose we have the special case that the recovery rate $\delta = 0$. Then $D(t_0, t, 0) = \exp(-\int_{t_0}^{t} r(s)ds)\exp(-\int_{t_0}^{t} \lambda(s)ds)$ and we see that $\lambda(t)$ has an interpretation as the spread between the yield on a risk-free zero coupon bond and the yield on a risky zero coupon bond i.e. $\lambda(t)$ is esssentially a measure of the credit-spread.

- This is intuitive but, at the same time, demonstrates a major shortcoming in our model. Except at the time of default, the credit-spread is deterministic in our model. In practice, credit-spreads demonstrate considerable volatility.

- In order to capture volatility in credit-spreads, we need to make $\lambda(t)$ a stochastic process. This leads to the idea of Cox processes.

- A Cox process is, essentially, a Poisson process with a stochastic intensity rate $\lambda(t)$. We need $\lambda(t)$ to be non-negative at all times. A possible process for $\lambda(t)$ is the CIR/Heston square-root process (other choices are possible).

- As before, we are interested in the time of the first jump of the Poisson process $N(t)$. Let τ be the (random) time to default, with $\tau > t_0 \equiv 0$. Then τ is modelled in the following way:

$$\begin{aligned}\mathrm{Prob}(\tau > t) &= \mathrm{Prob}(N(t) = 0) = \mathbb{E}_{t_0}^{\mathbb{Q}}[\mathbb{E}_{t_0}^{\mathbb{Q}}[\mathrm{Prob}(N(t) = 0)|\lambda]] \\ &= \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-\int_{t_0}^{t} \lambda(s)ds)],\end{aligned}$$

and where, in the first line, we have written $|\lambda$ to mean that we condition the expectation on the whole path of $\lambda(s)$ from t_0 to t. More mathematically, we are conditioning on the filtration generated by $\lambda(s)$ from t_0 to t, but, it does **NOT** include the filtration generated by $N(t)$ (else, we would certainly know whether default had occured by time t).

- Note that, if we define $\phi_\lambda(z)$ by $\phi_\lambda(z) = \mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz\int_{t_0}^{t}\lambda(s)ds)]$, we can rewrite the last equation as:

$$\mathrm{Prob}(\tau > t) = \phi_\lambda(i).$$

- In other words, the probability of no default between t_0 and t is simply the characteristic function of the process $\int_{t_0}^{t}\lambda(s)ds$, evaluated at $z = i$.

- If $\lambda(t)$ follows a CIR/Heston square-root process, then the characteristic function $\phi_\lambda(z)$ is known analytically.

- Furthermore, $\phi_\lambda(z)$ is also known analytically when $\lambda(t)$ follows several other types of non-negative processes.

- We can repeat our previous analysis to obtain a formula for the price of a risky bond in the Cox process model:

- We have a zero-coupon risky bond which promises to pay one at maturity t. However, in the event of default, at or before time t, it only pays δ, which again is a constant with $\delta < 1$, at maturity t. In the absence of arbitrage, the price $D(t_0, t, \delta)$, at time t_0, of the risky bond must be:

$$\begin{aligned} D(t_0, t, \delta) &= \mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(-\int_{t_0}^{t} r(s)ds) \left\{\text{Prob}(\tau > t)1 + (1 - \text{Prob}(\tau > t))\delta\right\}] \\ &= \exp(-\int_{t_0}^{t} r(s)ds) \left\{\delta + (1-\delta)\phi_\lambda(i)\right\}. \end{aligned}$$

- Modelling with jumps introduces some additional mathematical complexity.
- As a general rule, using Ito's lemma, Girsanov's theorem or trying to directly solve a PDE (or PIDE, more accurately) are less fruitful methodologies than in the pure diffusion case.
- The use of characteristic functions (and, more specifically, using them in connection with Fourier and Laplace transform methods) **is** extremely fruitful - indeed the key method for pricing options with jump processes.
- Monte Carlo simulation is also very useful - the books by Glasserman (Glasserman (2004)) , Schoutens (Schoutens (2003)) and Cont and Tankov (Cont and Tankov (2004)) all provide an excellent introduction to this.

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