Commodity options optimised

*In 2005, **John Crosby** introduced a very flexible framework in which it is possible to price derivatives, including exotics, on almost any underlying commodity. In this article, he shows how pricing can be done approximately 30 to 400 times faster than the methodology originally proposed*

In a recent article Crosby (2005), we introduced an arbitrage-free, multi-factor, jump-diffusion model for pricing commodity derivatives that appears flexible enough to model derivatives on almost any type of commodity. The model is consistent with any initial term structure of futures (or forward) commodity prices and captures stylised empirical observations made about commodity prices (for example, mean reversion, stochastic convenience yields and jumps). The model allows for multiple jump processes and also allows, when there are jumps, for the prices of long-dated futures contracts to jump by smaller amounts than short-dated contracts. This latter point is particularly noteworthy because, despite this being an empirically well-recognised feature of many commodity (especially natural gas and electricity) markets, it did not, to our knowledge, seem to have appeared in the literature before. Crosby (2005) shows that exotic commodity derivatives can be priced within this model using Monte Carlo simulation and also suggests that it is possible to calibrate the model parameters by deriving implied parameters from the market prices of options. This would rely on being able to rapidly calculate the prices of standard options. A methodology was proposed to compute the prices of standard options, which, in general, used a multi-dimensional Monte Carlo integration over the arrival times of the Poisson jumps (MCIATJ).

A number of authors have used Fourier transform (FT) methodology to price options, including Carr & Madan (1999), Heston (1993), Duffie, Pan & Singleton (2000)[1], Duffie, Filipovic & Schachermayer (2003), Lewis (2001), Sepp (2003) and Lee (2004). They have shown that very rapid computation of standard (and, indeed, in Duffie, Pan & Singleton (2000), also some exotic) European-style option prices is possible provided the relevant characteristic function is known in analytical form. In our model, the characteristic function, in general, does not have an analytical form. The aim of this article is to examine the possibility of pricing standard options utilising a FT methodology. We show this is indeed possible and is much faster (typically approximately 30 to 400 times faster) and more accurate than the MCIATJ methodology. The Fourier Transform Power Series (FTPS) methodology described in this article relies on a rapidly convergent power series expansion of terms appearing in the characteristic function of the terminal futures price distribution.

The rest of this article is organised as follows. First, we introduce notation and provide a very brief overview of the model. We then write down the form of the relevant characteristic function and derive a specific option price formula. Following that, we provide some numerical results and comparisons. We then conclude.

Overview of the model

In this article, we will work exclusively in the equivalent martingale measure (EMM), in which futures commodity prices are martingales, which, depending on the form of the model, may not be unique. In essence, in the case of non-uniqueness (which corresponds to market incompleteness[2]) we assume that an EMM has been 'fixed' through the market prices of options and, by an abuse of language, call this the (rather than an) EMM.

We denote expectations, at time t, with respect to the EMM by $E_t[]$. We will use the same notation as in Crosby (2005). We denote the (continuously compounded) risk-free short rate, at time t, by $r(t)$ and we denote the price, at time t, of a (credit risk-free) zero-coupon bond maturing at time T by $P(t, T)$.

We assume that interest rates are stochastic and follow a Gaussian interest rate model (for example, extended Vasicek, Babbs (1990), Hull & White (1993)), which is an arbitrage-free model consistent with any initial term structure of interest rates. The dynamics of bond prices under the EMM are (Babbs (1990) and Heath, Jarrow & Morton (1992)):

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sigma_P(t,T)\,dz_P(t)$$

where $\sigma_P(t, T)$ is a purely deterministic function of t and T, with $\sigma_P(T, T) = 0$, and $dz_P(t)$ denotes standard Brownian increments. We will provide numerical examples below for a one-factor Gaussian (extended Vasicek) model in which we write $\sigma_P(t, T) \equiv \sigma_r(1 - \exp(-\alpha_r(T - t)))/\alpha_r$, where σ_r and α_r are positive constants. However, all results in this article are applicable to any multi-factor Gaussian interest rate model without further ado.

We denote the futures commodity price, at time t, to (that is, the futures contract matures at) time T by $H(t, T)$. Our model is an arbitrage-free model consistent with any initial term structure of futures

[1] *Duffie, Pan & Singleton (2000) consider the family of affine jump-diffusion models. It can be shown that our model is also a member of this family of models*

[2] *For the important issue of hedging within a jump-diffusion model, see, for example, Hoogland, Neumann & Vellekoop (2001), Rebonato (2004) and Cont, Tankov & Voltchkova (2005) and the references therein*

commodity prices in which the dynamics of futures commodity prices under the EMM are:

$$\frac{dH(t,T)}{H(t-,T)} = \sum_{k=1}^{K} \sigma_{Hk}(t,T)\,dz_{Hk}(t) - \sigma_P(t,T)\,dz_P(t)$$
$$+\sum_{m=1}^{M}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)\right)-1\right)dN_{mt} - \sum_{m=1}^{M} e_m(t,T)\,dt \quad (1)$$

where:

$$e_m(t,T) \equiv \lambda_m(t)\,E_{Nmt}\left(\exp\left(\gamma_{mt}\exp\left(-\int_t^T b_m(u)\,du\right)\right)-1\right) \quad (2)$$

where, for each $k = 1, 2, \ldots, K$, $\sigma_{Hk}(t, T)$ are purely deterministic functions of at most t and T, $dz_{Hk}(t)$, for each k, are standard Brownian increments (which can be correlated with each other and with $dz_P(t)$ but we assume the correlations form a positive semi-definite correlation matrix), and N_{mt}, for each $m = 1, \ldots, M$, are independent Poisson processes whose intensity rates, under the EMM, at time t, are $\lambda_m(t)$, which are positive deterministic functions of at most t. The functions $b_m(t)$, for each m, are non-negative deterministic functions that we call jump decay coefficient functions. The parameters γ_{mt}, for each m, are parameters, which we call spot jump amplitudes. For each m, E_{Nmt} denotes the expectation operator, at time t, conditional on a jump occurring in N_{mt}.

We assume that the spot jump amplitudes γ_{mt} are one of two possible forms, which we term those of assumption 1 and assumption 2, which in turn are linked to two possible specifications of the jump decay coefficient functions $b_m(t)$.

For each $m = 1, \ldots, M$, we assume that either:

■ **Assumption 1**. The spot jump amplitude γ_{mt} is assumed to be a constant, which we denote by β_m. In this case, the jump decay coefficient function $b_m(t)$ is assumed to be any non-negative deterministic function, or:

■ **Assumption 2.** The spot jump amplitudes γ_{mt} are assumed to be independent and identically distributed random variables, each of which is independent of each of the Brownian motions and of each of the Poisson processes. In this case, the jump decay coefficient function $b_m(t)$ is assumed to be identically equal to zero, that is, $b_m(t) = 0$ for all t. Further, as in section 5 of Crosby (2005), we assume that, for this m, the spot jump amplitudes are normally distributed with mean β_m and standard deviation υ_m.

We define the indicator functions, for each $m = 1, \ldots, M$, $1_{m(1)} = 1$ if assumption 1 is satisfied, for this m, and $1_{m(1)} = 0$ otherwise and $1_{m(2)} = 1$ if assumption 2 is satisfied, for this m, and $1_{m(2)} = 0$ otherwise. Then equation (2) and assumptions 1 and 2 imply that:

$$\sum_{m=1}^{M} e_m(t,T) = \sum_{m=1}^{M}\left(1_{m(1)}\lambda_m(t)\left(\exp\left(\beta_m\exp\left(-\int_t^T b_m(u)\,du\right)\right)-1\right)\right)$$
$$+\sum_{m=1}^{M}\left(1_{m(2)}\lambda_m(t)\left(\exp\left(\beta_m+\frac{1}{2}\upsilon_m^2\right)-1\right)\right)$$

where we have used $b_m(t) \equiv 0$ if $1_{m(2)} = 1$.

Characteristic function of log futures price and an option price formula

Crosby (2005) shows that, within this model, it is possible to calculate the prices of standard European-style options (using the MCIATJ methodology). With this methodology, computation times are relatively fast for very short-dated options but they increase with increasing option maturity. We show in this article that our FTPS methodology is much faster than the MCIATJ methodology and does not suffer from this disadvantage.

Consider a standard European-style option, maturing at time T_1, written on a futures contract maturing at time T_2. The payout of the option at time T_1 is $\max(\eta(H(T_1, T_2) - K_{str}, 0))$ where K_{str} is the strike of the option and where $\eta = 1$ if the option is a call and $\eta = -1$ if it is a put. (Note $T_1 \le T_2$.)

We will use the Fourier methods of Lewis (2001) and Sepp (2003) to obtain the price of this option, at time t. First, we need the characteristic function of $\ln(H(T_1, T_2)/H(t, T_2))$. By direct calculation (and using some results in section 4 of Crosby, 2005), it is given by:

$$\phi_{T_1}(-z) \equiv E_t\left[\exp\left(-iz\ln\left(H(T_1,T_2)/H(t,T_2)\right)\right)\right]$$
$$= \exp\left(\frac{1}{2}\left(iz - z^2\right)\Sigma^2(t,T_1,T_2)\right)$$
$$\times\exp\left(iz\sum_{m=1}^{M}\int_t^{T_1} e_m(s,T_2)\,ds\right)\exp\left(-\sum_{m=1}^{M}\int_t^{T_1}\lambda_m(s)\,ds\right)$$
$$\times\exp\left(\sum_{m=1}^{M} 1_{m(1)}\left\{\int_t^{T_1}\lambda_m(s)\exp\left(-iz\beta_m\exp\left(-\int_s^{T_2} b_m(v)\,dv\right)\right)ds\right\}\right) \quad (3)$$
$$\times\exp\left(\sum_{m=1}^{M} 1_{m(2)}\left\{\left(\int_t^{T_1}\lambda_m(s)\,ds\right)\exp\left(-iz\beta_m - \frac{1}{2}\nu_m^2 z^2\right)\right\}\right)$$

where:

$$\Sigma^2(t,T_1,T_2) \equiv \int_t^{T_1} Var\left(\sum_{k=1}^{K}\sigma_{Hk}(s,T_2)\,dz_{Hk}(s) - \sigma_P(s,T_2)\,dz_P(s)\right)ds$$

Note this is non-negative.

We now calculate the option price using results in Lewis (2001) and Sepp (2003).

■ **Proposition 1.** Define:

$$A(s,T_1,T_2) \equiv \mathrm{cov}\left(\frac{dP(s,T_1)}{P(s,T_1)}, \frac{dH(s,T_2)}{H(s-,T_2)}\right) \quad \text{and}$$
$$X \equiv \frac{K_{str}}{\exp\left(\int_t^{T_1} A(s,T_1,T_2)\,ds\right)} \quad (4)$$

Then the price, at time t, of the standard European-style option, maturing at time T_1, written on a futures contract maturing at time T_2, where $t \le T_1 \le T_2$, is:

$$E_t\left[\exp\left(-\int_t^{T_1} r(s)\,ds\right)\max\left(\eta\left(H(T_1,T_2) - K_{str}, 0\right)\right)\right]$$
$$= \left(\frac{1+\eta}{2}\right)H(t,T_2)P(t,T_1)\exp\left(\int_t^{T_1} A(s,T_1,T_2)\,ds\right)$$
$$+\left(\frac{1-\eta}{2}\right)K_{str}P(t,T_1) \quad (5)$$
$$-P(t,T_1)\varpi(t,M,T_1,T_2)\int_0^{\infty}\Omega(u;M,T_1,T_2)\Theta\left(u;\Sigma^2(t,T_1,T_2)\right)du$$

See footnote 3 for an explanation of notation.

■ **Proof.** We can use Girsanov's theorem to effectively 'pull' the discounting term:

$$\exp\left(-\int_t^{T_1} r(s)\,ds\right)$$

outside of the expectation. This produces the factors involving:

$$\int_t^{T_1} A(s,T_1,T_2)\,ds$$

The rest of the proof follows by substituting our characteristic function (equation 3) into equation 3.11 of Lewis (2001) or, equivalently, into equation 3.18 of Sepp (2003) and simplifying.

It is possible, by the use of numerical integration techniques, to evaluate equation (5) and hence calculate the prices of standard European-style options. This is straightforward and efficient if all the Poisson processes satisfy assumption 2 and the intensity rates $\lambda_m(s)$ are easily integrable. However, if any of the Poisson processes satisfy assumption 1, the necessity to calculate the integrals $J_2(u, m, T_1, T_2)$ and $J_3(u, m, T_1, T_2)$ with respect to s, which are inside the integral with respect to u, makes this quite computationally intensive. In addition, the term $J_1(m, T_1, T_2)$ will also need to be calculated by numerical integration (although this is less problematic since it can be calculated once outside the integral with respect to u and stored). Therefore, we look for a methodology, which will allow option prices to be calculated much more quickly and more accurately than a direct numerical (double) integration as in equation (5).

■ **Assumption 3.** We will henceforth assume that, for each m, $m = 1, \ldots, M$, $\lambda_m(s) \equiv \lambda_m$ and $b_m(s) \equiv b_m$ are constants.

We will assume in what follows that, for each $m = 1, \ldots, M$, $\beta_m \neq 0$ and $b_m > 0$. These conditions are not restrictive since in the first case there would (if $\beta_m = 0$) be no jumps and in the second case $b_m = 0$ means that the problem becomes a special case of assumption 2, which is much simpler. Now we will show that option prices can be more readily calculated as follows.

Proposition 2 is inspired by results in Abramowitz & Stegun (1970).

■ **Proposition 2.** Define $sign(\beta_m) = \beta_m/|\beta_m|$ where $|\beta_m|$ is the absolute value of β_m. Define $\psi_1 = |\beta_m|\exp(-b_m(T_2 - t))$ and $\psi_2 = |\beta_m|\exp(-b_m(T_2 - T_1))$. Define $y = |\beta_m|\exp(-b_m(T_2 - s))$. Define θ via $0 \leq \theta \leq \pi/2$ and $\theta \equiv \tan^{-1}(2u)$, for $u \geq 0$. Then:

$$\begin{aligned} J_1(m,T_1,T_2) &\equiv \lambda_m \int_t^{T_1}\left(\exp\left(\beta_m \exp\left(-b_m\left(T_2-s\right)\right)\right)-1\right)ds \\ &= \frac{\lambda_m}{b_m}\left(\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{1}{n!}\left(\beta_m \exp\left(-b_m\left(T_2-T_1\right)\right)\right)^n \right.\right.\\ &\qquad \left.\left. \times\left(1-\exp\left(-b_m\left(T_1-t\right)n\right)\right)\right]\right) \end{aligned} \tag{7}$$

$$\begin{aligned} \Omega\left(u;M,T_1,T_2\right) \equiv \cos&\left[u\left\{\ln\left(\frac{H(t,T_2)}{X}\right)\right\} - u\left\{\sum_{m=1}^{M}1_{m(1)}\left\{J_1\left(m,T_1,T_2\right)\right\}\right\}\right. \\ &-u\left\{\sum_{m=1}^{M}1_{m(2)}\left\{\left(\int_t^{T_1}\lambda_m(s)\,ds\right)\left(\exp\left(\beta_m+\frac{1}{2}\nu_m^2\right)-1\right)\right\}\right\} + \left(\sum_{m=1}^{M}1_{m(1)}\left\{J_2\left(u,m,T_1,T_2\right)\right\}\right) \\ &\left.+\left(\sum_{m=1}^{M}1_{m(2)}\left\{\exp\left(\frac{1}{2}\beta_m+\frac{1}{8}\nu_m^2-\frac{1}{2}\nu_m^2u^2\right)\sin\left(u\left(\beta_m+\frac{1}{2}\nu_m^2\right)\right)\left(\int_t^{T_1}\lambda_m(s)\,ds\right)\right\}\right)\right] \\ \times\exp&\left(\sum_{m=1}^{M}1_{m(2)}\left\{\exp\left(\frac{1}{2}\beta_m+\frac{1}{8}\nu_m^2-\frac{1}{2}\nu_m^2u^2\right)\cos\left(u\left(\beta_m+\frac{1}{2}\nu_m^2\right)\right)\left(\int_t^{T_1}\lambda_m(s)\,ds\right)\right\}\right)\exp\left(\sum_{m=1}^{M}1_{m(1)}\left\{J_3\left(u,m,T_1,T_2\right)\right\}\right), \end{aligned}$$

$$\Theta\left(u;\Sigma^2\left(t,T_1,T_2\right)\right) \equiv \left(\exp\left(-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\Sigma^2\left(t,T_1,T_2\right)\right)\Big/\left(u^2+\frac{1}{4}\right)\right),$$

$$\begin{aligned} \varpi\left(t,M,T_1,T_2\right) &\equiv \frac{1}{\pi}\exp\left(\int_t^{T_1}A\left(s,T_1,T_2\right)ds\right)\sqrt{XH\left(t,T_2\right)}\exp\left(-\frac{1}{2}\sum_{m=1}^{M}1_{m(1)}\left\{J_1\left(m,T_1,T_2\right)\right\}\right) \\ &\times\exp\left(-\frac{1}{2}\sum_{m=1}^{M}1_{m(2)}\left\{\left(\int_t^{T_1}\lambda_m(s)\,ds\right)\left(\exp\left(\beta_m+\frac{1}{2}\nu_m^2\right)-1\right)\right\}\right)\exp\left(-\sum_{m=1}^{M}\int_t^{T_1}\lambda_m(s)\,ds\right), \end{aligned} \tag{6}$$

$$J_1\left(m,T_1,T_2\right) \equiv \int_t^{T_1}\lambda_m(s)\left(\exp\left(\beta_m\exp\left(-\int_s^{T_2}b_m(v)\,dv\right)\right)-1\right)ds,$$

$$J_2\left(u,m,T_1,T_2\right) \equiv \int_t^{T_1}\exp\left(\frac{1}{2}\beta_m\exp\left(-\int_s^{T_2}b_m(v)\,dv\right)\right)\sin\left(u\beta_m\exp\left(-\int_s^{T_2}b_m(v)\,dv\right)\right)\lambda_m(s)\,ds,$$

$$J_3\left(u,m,T_1,T_2\right) \equiv \int_t^{T_1}\exp\left(\frac{1}{2}\beta_m\exp\left(-\int_s^{T_2}b_m(v)\,dv\right)\right)\cos\left(u\beta_m\exp\left(-\int_s^{T_2}b_m(v)\,dv\right)\right)\lambda_m(s)\,ds$$

$$J_3\left(u,m,T_1,T_2\right) \equiv \int_t^{T_1} \exp\left(\frac{1}{2}\beta_m \exp\left(-b_m\left(T_2-s\right)\right)\right)$$
$$\times\cos\left(u\beta_m \exp\left(-b_m\left(T_2-s\right)\right)\right)\lambda_m ds = \lambda_m\left(T_1-t\right) \tag{8}$$
$$+\frac{\lambda_m}{b_m}\left(\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{\left(sign\left(\beta_m\right)\sqrt{4u^2+1}\right)^n\left(\psi_2{}^n-\psi_1{}^n\right)\cos\left(n\theta\right)}{2^n n!}\right]\right)$$

$$J_2\left(u,m,T_1,T_2\right) \equiv \int_t^{T_1} \exp\left(\frac{1}{2}\beta_m \exp\left(-b_m\left(T_2-s\right)\right)\right)$$
$$\times\sin\left(u\beta_m \exp\left(-b_m\left(T_2-s\right)\right)\right)\lambda_m ds \tag{9}$$
$$=\frac{\lambda_m}{b_m}\left(\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{\left(sign\left(\beta_m\right)\sqrt{4u^2+1}\right)^n\left(\psi_2{}^n-\psi_1{}^n\right)\sin\left(n\theta\right)}{2^n n!}\right]\right)$$

■ **Proof.** Consider $J_3(u, m, T_1, T_2) + iJ_2(u, m, T_1, T_2)$. By combining sines and cosines into the complex exponential function:

$$J_3\left(u,m,T_1,T_2\right)+iJ_2\left(u,m,T_1,T_2\right)$$
$$=\int_t^{T_1}\exp\left(\frac{1}{2}\beta_m\exp\left(-b_m\left(T_2-s\right)\right)\right)\exp\left(iu\beta_m\exp\left(-b_m\left(T_2-s\right)\right)\right)\lambda_m ds$$

By substitution for y and θ and then by power series expansion of the exponential function, the above is equal to:

$$\frac{\lambda_m}{b_m}\int_{\psi_1}^{\psi_2}\frac{\exp\left(sign\left(\beta_m\right)\left(\cos\theta+i\sin\theta\right)y\sqrt{u^2+\frac{1}{4}}\right)}{y}dy$$

$$=\frac{\lambda_m}{b_m}\left(\int_{\psi_1}^{\psi_2}\left\{\left[\frac{1}{y}+\sum_{n=1}^{\infty}\frac{1}{y}\frac{\left(sign\left(\beta_m\right)\sqrt{u^2+\frac{1}{4}}\right)^n\left(\cos\theta+i\sin\theta\right)^n y^n}{n!}\right]\right\}dy\right)$$

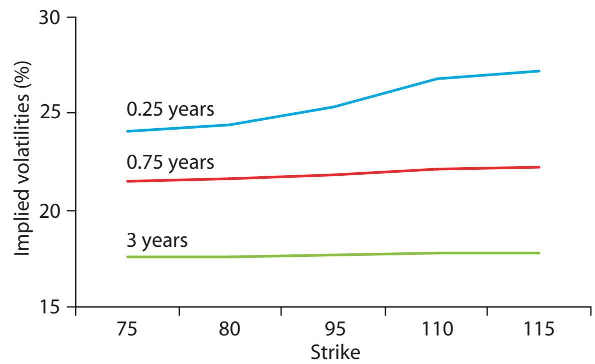
$$=\lambda_m\left(T_1-t\right)+$$
$$\frac{\lambda_m}{b_m}\left(\sum_{n=1}^{\infty}\left[\frac{1}{n}\frac{\left(sign\left(\beta_m\right)\sqrt{4u^2+1}\right)^n\left(\psi_2^n-\psi_1^n\right)\left(\cos\left(n\theta\right)+i\sin\left(n\theta\right)\right)}{2^n n!}\right]\right)$$

by performing the integral term by term and simplifying. Now taking the real and imaginary parts yields equations (8) and (9). The proof of equation (7) is very similar and therefore omitted.

Using these power series expressions is much faster than numerical integration. We can evaluate a sufficient number of terms to calculate the integrals in equations (7), (8) and (9) to any desired tolerance. We can then, substituting equations (7), (8) and (9) into equation (5), calculate option prices using a single one-dimensional integration irregardless of how many Brownian motions and Poisson processes drive the futures commodity prices.

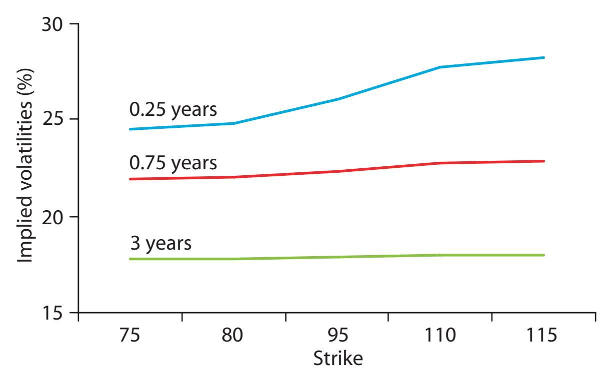
We should make a further comment about the convergence of the integral in equation (5). It is clear that the dominant term in the integrand as $u \to \infty$ is $\Theta(u; \Sigma^2(t, T_1, T_2))$, which clearly tends to zero as $u \to \infty$. Clearly, convergence will be faster when $\Sigma^2(t, T_1, T_2)$ is large. Conversely, when $\Sigma^2(t, T_1, T_2)$ is small, then convergence will be slower. This will happen, for example, when the time to maturity of the option is small.

1 Implied Black volatilities, with the FTPS methodology, for example 1



Note: only three option maturities are shown for clarity

2 Implied Black volatilities, with the FTPS methodology, for example 2



Note: only three option maturities are shown for clarity

Numerical examples

Here, we will compare the results of our FTPS methodology with the results using the MCIATJ methodology by providing two numerical examples, labelled 1 and 2, the results of which are in tables A and B respectively.

For the MCIATJ methodology, we proceed exactly as in Crosby (2005) and, as there, we use 1,500 Monte Carlo simulations and report the standard errors of the option prices. For our FTPS methodology, we evaluate the integral with respect to u in equation (5) using Simpson's rule with 1,024 points. In view of the term $\Theta(u; \Sigma^2(t, T_1, T_2))$ in equations (5) and (6), we truncate the upper limit of the integral when the value of u is such that $\Theta(u; \Sigma^2(t, T_1, T_2))$ is less then 10^{-8}. We truncate the infinite series in equations (7), (8) and (9) when the value of an additional term in the series has converged to less than 10^{-12}.

In our FTPS methodology, we must use an heuristic rule to esti-

A Example 1: one Poisson process

Option prices using the MCIATJ methodology					
	75	80	95	110	115
0.25	19.8460	15.1892	4.7491	0.9345	0.5129
0.50	19.9199	15.6447	6.0987	1.7881	1.1347
0.75	19.9956	15.9661	6.9049	2.4148	1.6419
1	20.0410	16.1943	7.4844	2.9143	2.0654
2	20.0639	16.7238	8.9826	4.3986	3.4127
3	19.9732	16.9906	9.9626	5.5164	4.4828
Standard errors for the option prices above					
	75	80	95	110	115
0.25	< 0.0001	< 0.0001	< 0.0001	< 0.0001	< 0.0001
0.50	< 0.0001	< 0.0001	0.0001	0.0003	0.0004
0.75	0.0001	0.0002	0.0005	0.0008	0.0009
1	0.0003	0.0004	0.0009	0.0014	0.0013
2	0.0009	0.0012	0.0019	0.0025	0.0026
3	0.0011	0.0014	0.0021	0.0028	0.0028
Option prices using the FTPS methodology					
	75	80	95	110	115
0.25	19.8460	15.1892	4.7491	0.9344	0.5129
0.50	19.9199	15.6447	6.0986	1.7881	1.1347
0.75	19.9956	15.9660	6.9050	2.4147	1.6410
1	20.0410	16.1943	7.4838	2.9147	2.0667
2	20.0645	16.7226	8.9833	4.3986	3.4120
3	19.9731	16.9901	9.9630	5.5139	4.4833
Heuristic error estimates for the option prices above using the FTPS methodology					
	75	80	95	110	115
0.25	7.78E-08	–6.09E-07	–3.99E-07	–4.14E-07	1.48E-07
0.50	–6.52E-08	2.80E-08	–3.43E-09	–7.80E-08	9.32E-09
0.75	–5.19E-08	6.23E-08	–1.76E-08	–6.61E-08	–6.83E-08
1	–2.61E-08	7.39E-08	–3.54E-08	–2.32E-08	–8.90E-08
2	8.76E-09	2.04E-08	–1.71E-08	1.92E-08	–1.02E-09
3	2.16E-08	2.25E-08	–2.56E-08	3.13E-08	2.32E-08

Note: all options are standard European-style calls on futures. The strikes of the options are across the first row. The values of T_1 are down the first column. In each case, the time to maturity of the underlying futures contract is given by $T_2 = T_1 + 0.125$, that is, the futures contract matures 0.125 years after the maturity of the option

B Example 2: three Poisson processes

Option prices using the MCIATJ methodology					
	75	80	95	110	115
0.25	19.8554	15.2171	4.8723	1.0370	0.5913
0.50	19.9521	15.7049	6.2423	1.9175	1.2436
0.75	20.0449	16.0451	7.0592	2.5582	1.7671
1	20.1023	16.2847	7.6420	3.0649	2.2002
2	20.1408	16.8206	9.1260	4.5375	3.5451
3	20.0465	17.0788	10.0822	5.6348	4.6004
Standard errors for the option prices above					
	75	80	95	110	115
0.25	< 0.0001	< 0.0001	< 0.0001	< 0.0001	< 0.0001
0.5	< 0.0001	< 0.0001	< 0.0001	0.0002	0.0002
0.75	< 0.0001	0.0001	0.0003	0.0005	0.0005
1	0.0002	0.0002	0.0005	0.0007	0.0008
2	0.0004	0.0006	0.0009	0.0015	0.0011
3	0.0005	0.0006	0.0009	0.0011	0.0011
Option prices using the FTPS methodology					
	75	80	95	110	115
0.25	19.8554	15.2171	4.8723	1.0370	0.5913
0.50	19.9521	15.7049	6.2423	1.9176	1.2439
0.75	20.0450	16.0451	7.0592	2.5584	1.7672
1	20.1023	16.2849	7.6423	3.0653	2.2020
2	20.1410	16.8209	9.1265	4.5404	3.5453
3	20.0462	17.0788	10.0826	5.6349	4.5996
Heuristic error estimates for the option prices using the FTPS methodology					
	75	80	95	110	115
0.25	1.15E-08	–5.70E-07	–2.86E-07	–2.94E-07	1.08E-07
0.50	–5.71E-08	4.18E-08	2.00E-08	–5.93E-08	3.23E-08
0.75	–2.49E-08	5.82E-08	1.02E-08	–6.54E-08	–3.59E-08
1	3.08E-10	5.81E-08	–4.90E-09	–3.99E-08	–6.72E-08
2	9.74E-09	1.37E-08	–9.44E-09	1.08E-08	–5.45E-09
3	1.85E-08	1.53E-08	–1.77E-08	2.41E-08	1.47E-08

mate the errors. Our approach is to revalue the same options using Simpson's rule with 8,192 points (eight times more than before), truncating the upper limit of the integral in equation (5) when the value of u is such that $\Theta(u; \Sigma^2(t, T_1, T_2))$ is less then 10^{-9} (one tenth of the previous tolerance) and truncating the infinite series in equations (7), (8) and (9) when the absolute value of an additional term in the series has converged to less than 10^{-13} (one tenth of the previous tolerance). We then report the differences (labelled heuristic error estimates in tables A and B) between the original option price estimates and these improved option price estimates. Of course, this rule is heuristic and these heuristic error estimates are not strictly comparable with the standard errors of the MCIATJ methodology but comparing the two should give approximate estimations of the relative accuracies of the two different methodologies.

Computations were performed on a desktop personal computer, running at 2.8GHz, with Microsoft Windows 2000 Professional, with 1Gb of RAM with a program written in Microsoft C++.

In both examples 1 and 2, we assume that the futures commodity prices for all maturities are 95, the interest rate yield curve is flat with a continuously compounded risk-free rate of 0.05 (as in Miltersen & Schwartz (1998) and Crosby (2005)) and that interest rates follow a one-factor extended Vasicek model in which $\sigma_p(t, T) \equiv \sigma_r(1 - \exp(-\alpha_r(T - t)))/\alpha_r$, where $\sigma_r = 0.0096$ and $\alpha_r = 0.2$.

We use the same diffusion parameters as in Miltersen & Schwartz (1998) and Crosby (2005), that is, we have two Brownian motions (in addition to the Brownian motion driving interest rates) that is, $K = 2$ and, for each k, $\sigma_{Hk}(t, T) \equiv \eta_{Hk} + \chi_{Hk} \exp(-a_{Hk}(T - t))$, with $\eta_{H1} = 0.266$, $\eta_{H2} = 0.249/1.045 \approx 0.23827751196$, $\chi_{H1} = 0.0$, $\chi_{H2} = -0.249/1.045$, $a_{H2} = 1.045$, $correl(dz_{H1}(t), dz_{H2}(t)) = -0.805$, $correl(dz_{H1}(t), dz_p(t)) = -0.0964$, $correl(dz_{H2}(t), dz_p(t)) = 0.1243$.

In both examples 1 and 2, we value standard European-style call options on futures contracts whose maturities are 0.125 years after the maturity of the option. We price options with strikes 75, 80, 95, 110 and 115, and option maturities equal to 0.25, 0.5, 0.75, one, two, and three years (there are 30 options in total).

■ **Example 1.** Here, we assume there is one Poisson process, $M = 1$, it satisfies assumption 1 and it has parameters (which are purely for illustration) $\lambda_1 = 0.75$, $\beta_1 = 0.22$ and $b_1 = 2.0$.

We price the 30 options described above. The results are shown in table A. As in Crosby (2005), the calculation time for all 30 options using the MCIATJ methodology was 0.51 seconds. Using our FTPS methodology, the calculation time for all 30 options is less than 0.016 seconds, corresponding to an average of approximately 0.00053 seconds per option. The FTPS methodology is, on average, approximately 32 times faster than the MCIATJ methodology. Comparing the heuristic error estimates (the largest, in absolute value, of which is 0.000000609) with the standard errors reported in Crosby (2005) (which vary between less than 0.0001 and 0.0028) suggests that the accuracy of the FTPS methodology is much better than that of the MCIATJ methodology. More detailed analysis (not reported) of the results in table A shows that for options with a maturity of 0.25 years (the shortest maturity), the MCIATJ methodology is actually very competitive (in terms of both speed and accuracy) with the FTPS methodology. The benefits of the FTPS methodology are seen with options with longer times to maturity.

■ **Example 2.** Here, we assume there are three Poisson processes, M = 3, and they all satisfy assumption 1, with parameters (which, again, are purely for illustration) $\lambda_1 = 0.25$, $\beta_1 = 0.32$, $b_1 = 3.0$, $\lambda_2 = 0.30$, $\beta_2 = 0.22$, $b_2 = 2.0$, $\lambda_3 = 0.35$, $\beta_3 = 0.16$ and $b_3 = 1.0$.

We price the same 30 options as before. The results are shown in table B. The calculation time for all 30 options using the MCIATJ methodology was approximately 20.17 seconds. Using our FTPS methodology, the calculation time for all 30 options is approximately 0.0414 seconds or an average of approximately 0.00138 seconds per option. The FTPS methodology is, on average, approximately 487 times faster than the MCIATJ methodology. The largest (in absolute value) heuristic error estimate across the 30 options is 0.00000057. Comparing the heuristic error estimates with the standard errors in table B suggests, again, that the accuracy of the FTPS methodology is very much better than that of the MCIATJ methodology. The accuracy of the FTPS methodology is also reasonably uniform across the 30 different options, whereas the MCIATJ methodology is typically much more (respectively less) accurate at lower (respectively higher) strikes and shorter (respectively longer) times to option maturity.

In the examples above, each of the 30 option prices was calculated individually. Total calculation times could be reduced further, by exploiting the fact that most terms in equation (5) are independent of the strike of the option. A further possibility might be to use the fast Fourier transform (FFT) approach of Carr & Madan (1999) to obtain option prices across a wide range of strikes simultaneously. Unfortunately, equation (5) is not compatible with the FFT algorithm. However, Carr & Madan (1999) describes two approaches that are fully compatible with the FFT algorithm and which can be used with essentially arbitrary characteristic functions. It might be possible to use their FFT approach with the characteristic function of equation (3) and with power series expansions similar to equations (7), (8) and (9). However, we leave this for future research.

We have focused on pricing standard European-style options on futures but clearly, using results in Crosby (2005) and Miltersen & Schwartz (1998), it is equally possible to price standard options on forwards and futures-style options on futures. Indeed, the price of a futures-style option on futures (whether European-style or American-style) is simply given by equation (5) with $A(s, T_1, T_2)$ formally replaced by zero and $P(t, T_1)$ formally replaced by unity.

Conclusions

We have shown that it is possible to evaluate the prices of standard European-style options considerably more accurately and between approximately 30 to 400 times faster than the method originally proposed in Crosby (2005). Our FTPS methodology is extremely accurate, robust and straightforward to implement. This will prove very useful for calibrating the model parameters by deriving implied parameters from the market prices of commodity options. Once the model parameters have been calibrated, the Crosby (2005) model is a very flexible framework in which it is possible to price a portfolio of commodity derivatives (including exotics) on a consistent basis. ■

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