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Approximating Lévy processes by a hyperexponential jump-diffusion process with a view to option pricing

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Abstract

In this paper we work on how to approximate a Lévy process by a hyperexponential jump-diffusion process, which is composed of a Brownian motion and of an arbitrary number of sums of compound Poisson processes with double exponentially distributed jumps. This approximation will facilitate the pricing of exotic options since hyperexponential jump-diffusion processes have a degree of tractability that other Lévy processes do not have.

The idea of this approximation relies on Bernstein's theorem and has been applied to option pricing by Asmussen et al. (2007) and Jeannin and Pistorius (2008). In this dissertation we introduce a more systematic methodology for constructing this approximation which allow us to compute the intensity rates, the mean jump sizes and the volatility of the approximating hyperexponential process (almost) analytically. The methodology is very easy to implement.

We compute vanilla option prices and barrier option prices using the approximating hyperexponential process and we compare our results to those obtained from other methodologies in the literature. We demonstrate that our methodology gives very accurate option prices and that these prices are more accurate than those obtained from existing methodologies for approximating Lévy processes by hyperexponential jump-diffusion processes.

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Contents

1 Introduction

The purpose of this dissertation is to examine how a Lévy process can be approximated by a jump-diffusion process with jumps consisting of sums of compound Poisson processes with double exponentially distributed jumps. Firstly, we need to introduce some concepts and notation.

We define the initial time (today) by $t_0 \equiv 0$ and denote calendar time by t, $t \geq t_0$. Consider an economy, which we assume to be free of arbitrage, where the risk-free interest rate is r and in which there is an asset, which pays a dividend yield q, whose price at time t is S_t.

In the standard option-pricing theory, due to Black and Scholes (1973) and Merton (1973), under the risk-neutral equivalent martingale measure $\mathbb{Q}$, the asset price evolves as:

$$S_t = S_{t_0}\exp((r-q)t + X_t), \tag{1.1}$$

where X_t is a Brownian motion, with $X_{t_0} = 0$, with constant volatility σ, say, and drift term $-\sigma^2/2$. The process X_t has stationary and independent increments and furthermore, for any times t_1 and t_2, with $t_2 > t_1$, $X_{t_2} - X_{t_1}$ is normally distributed with mean $-\sigma^2(t_2 - t_1)/2$ and variance $\sigma^2(t_2 - t_1)$. These properties define Brownian motion. When this model was first introduced, it was accepted by market participants as the appropriate model with which to price options. However, at least since the time of the stock market crash of 1987, options markets have shown implied volatilities that display smiles and skews. This model is not capable of producing such behaviour. Consequently, researchers have looked for ways of modifying the standard model of Black and Scholes (1973) and Merton (1973) in order to capture the effect of smiles and skews in implied volatilities.

One such way is to replace the assumption that X_t is a Brownian motion by the assumption that X_t is a Lévy process. This still implies that X_t has stationary and independent increments but now $X_{t_2} - X_{t_1}$ no longer has to be normally distributed. These properties essentially define a Lévy process. Note that Brownian motion is also a Lévy process and it is the only Lévy process with continuous sample paths. Hence, all other Lévy processes must have jumps (ie discontinuous sample paths). This implies that the risk-neutral equivalent martingale measure is not unique. We will assume that one such measure $\mathbb{Q}$ has been fixed. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and an information filtration $\{\mathcal{F}_t\}_{t_0 \leq t < \infty}$ which we assume satisfies the usual conditions. We denote by $\mathbb{E}_t^{\mathbb{Q}}[-]$ the expectation operator, under $\mathbb{Q}$, at time t. Under the risk-neutral equivalent martingale measure $\mathbb{Q}$, the asset price evolves as:

$$S_t = S_{t_0}\exp((r-q)t + X_t), \tag{1.2}$$

where X_t is a Lévy process, with $X_{t_0} = 0$, which has been mean-corrected such that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$.

Examples of Lévy processes which have been used in finance include Variance Gamma (henceforth VG) (Madan et al. (1998)), CGMY (Carr et al. (2002), (2003)) (also known as the KoBol process, Boyarchenko and Levendorskii (2000), (2002)), Normal Inverse Gaussian (henceforth NIG) (Barndorff-Nielsen (1998)) and Generalised Hyperbolic (Eberlein et al. (1998), Eberlein and Prause (1998), (1999)). For more details on these processes, we refer the reader to Schoutens (2003) and Cont and Tankov (2004). All the aforementioned processes have infinite activity (that is to say there are an infinite number of jumps in a finite time interval, see Schoutens (2003) - actually, to be precise, in the CGMY model, the Y parameter must be non-negative to have infinite activity). We will refer to these aforementioned infinite activity Lévy processes by the collective noun General Classes of Lévy processes.

A class of processes which, in a loose sense, lie between Brownian motion and General Classes of Lévy processes are jump-diffusion processes. These processes (which are also types of Lévy processes in that they have stationary and independent increments) combine a Brownian motion component with one or more compound Poisson processes (and hence have finite activity - that is to say a finite number of jumps in a finite time interval).

We define the characteristic function by $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iz(\log(S_t/S_{t_0})))]$ for any (possibly complex) z. The characteristic function is known (in essentially closed form) for all the Lévy processes mentioned above. Results from Lipton (2001), Lewis (2001) and Sepp (2003) (see also section 5), show that given the characteristic function, it is straightforward to price vanilla (standard European) options very rapidly. However, pricing exotic options is much more complicated for General Classes of Lévy processes. Whereas analytical solutions exist for many simple exotic options in the model of Black and Scholes (1973) and Merton (1973), analytical results are rarely available for General Classes of Lévy processes. This means that pricing exotic options generally requires Monte Carlo simulation. There are techniques for simulating Lévy processes and we briefly review some of them in section 3. However, many of them, broadly-speaking, come down to approximating the Lévy process by a jump-diffusion process. Typically, very small jumps (of which there are an infinity number with an infinite activity Lévy process) are approximated by Brownian motion (a Central Limit Theorem type of argument provides an intuition for this but see Asmussen and Rosinski (2001) and Cont and Tankov (2004) for details about when this is fully justified). We remark that there are series representations (see Rosinski (2001), (2002), Schoutens (2003), Cont and Tankov (2004), Crosby (2007)) which do this approximation in very clever ways but they can still be viewed, broadly-speaking, as approximating the Lévy process by a jump-diffusion process.

There is a class of processes which does allow for analytical results (up to Laplace inversion) for a range of exotic options. These are jump-diffusion processes which have a Brownian motion component as well as a jump component formed from sums of compound Poisson processes with double exponentially distributed jumps (henceforth HEJD for hyperexponential jump-diffusion).

Kou and Wang (2003), (2004) and Sepp (2004) provide analytical results (up to Laplace inversion) for various types of barrier options under the Kou (2002) double exponential jump-diffusion (henceforth DEJD) process (see also Lipton (2002)). Working under this same process, Kou and Wang (2004) and Sepp (2005) provide analytical results (up to Laplace inversion) for lookback options and Russian options. Asmussen et al. (2007) and Jeannin and Pistorius (2008) price several type of single barrier options under a HEJD process which extends the Kou (2002) DEJD process by allowing for an arbitrary number of sums of compound Poisson processes. Carr and Crosby (2008) go yet further by providing analytical results (up to Laplace inversion) for double barrier options of several different types under a process they call the CEE2 process. This is a jump-diffusion process based on a HEJD process but, in addition, the diffusion volatility and the jump intensity rates can be stochastic and driven by a two-state continuous-time Markov chain (see also Asmussen et al. (2004) and Pistorius (2004)). Carr and Crosby (2008) also provide analytical results for first passage time distributions for CEE2 processes. Hence, we can see that a HEJD jump-diffusion process has, from the point of view of pricing exotic options, a considerable degree of tractability which General Classes of Lévy processes such as CGMY and NIG do not have. As an aside, we also mention that Monte Carlo simulation of HEJD processes (and of CEE2 processes), without discretization error, is very straightforward (see, for example, Glasserman (2004), p114 of Schoutens (2003) and Carr and Crosby (2008)).

It is known from the literature that any Lévy process with a Lévy density which is monotone, as one moves away from the origin, can be approximated (in distribution) by a HEJD process. At an intuitive level, this could be thought of as a fitting or splining-type procedure since HEJD processes also have a Lévy density which is monotone as one moves away from the origin. However, for Lévy processes with a completely monotone Lévy density, a more precise motivation comes from Bernstein's theorem. Let us denote the Lévy density by $\nu(x)$. By definition, the Lévy density $\nu(x)$ is completely monotone if, and only if, for all $k \geq 0$, $(-1)^k d\nu(x)/dx^k > 0$ (this is for $x > 0$; for $x < 0$, we replace $\nu(x)$ by $\nu(-x)$). It is easily verified that VG, NIG, Generalised Hyperbolic and CGMY (the latter provided the parameter $Y \geq -1$) processes all have Lévy densities which satisfy this condition. If this condition is satisfied, then Bernstein's theorem tells us (see also Jeannin and Pistorius (2008)) that we can write $\nu(x)$ in the form:

$$\nu(x) = \mathbf{1}_{\{x>0\}} \int_0^{+\infty} e^{-ux} \mu_+(u)\ du + \mathbf{1}_{\{x<0\}} \int_{-\infty}^{0} e^{-ux} \mu_-(u)\ du \tag{1.3}$$

where $\mu_-(u)$ and $\mu_+(u)$ are some measures on $(-\infty, 0)$ and $(0, \infty)$ respectively and $\mathbf{1}$ denotes the indicator function.

Observing the forms of the integrands and noting that an integral can be approximated as a discrete sum and approximating the infinite limits by finite quantities, this last equation immediately suggests approximating the Lévy process as a HEJD process (with jumps whose magnitudes are smaller than some certain level being approximated by Brownian motion). This idea was introduced into the option pricing literature by Asmussen et al. (2007) and then developed further by Jeannin and Pistorius (2008). Asmussen et al. (2007) considered the pricing of barrier options within the CGMY process by approximating the CGMY process by a HEJD process. Jeannin and Pistorius (2008) also considered approximating the VG process and the NIG process by a HEJD process and then proceeded to price barrier options as well as to obtain risk sensitivities (eg first and second partial derivatives of the option price with respect to the initial asset price).

The methodologies employed by Asmussen et al. (2007) and by Jeannin and Pistorius (2008) to approximate the Lévy process in question by a HEJD process are described in those papers in more depth. However, a brief precis is as follows:

They fitted a total of fourteen compound Poisson processes with exponentially distributed jumps, seven producing up jumps and seven producing down jumps. They chose (based on intuition) some points x at which to approximate the Lévy density $\nu(x)$. They chose (again, based on intuition) some mean jump sizes for the individual exponentially-distributed jumps which constituted the HEJD process. In Asmussen et al. (2007), they did a non-linear least-squares fit, over the choice of the mean jump sizes and the jump intensity rates, between the Lévy density $\nu(x)$ and the Lévy density of the HEJD process evaluated at the chosen points x. In Jeannin and Pistorius (2008), they kept the mean jump sizes the same as the initial guesses and only fitted the jump intensity rates in the least-squares fit. The diffusion component was determined by approximating as Brownian motion all the jumps whose magnitude were less than the smallest mean jump sizes (for both up and down jumps).

One might describe these fitting procedures as a little ad-hoc. This is not a criticism. Indeed, firstly, the approximation of the Lévy density was by no means the central point of either of those papers, secondly, the fitting procedures are intuitive and easy to implement and, thirdly, based on results reported there and also based on results contained within this dissertation (see section 6), the resulting barrier option prices are reasonably accurate. However, the question might still be asked as to whether there might be an alternative methodology. By contrast with Asmussen et al. (2007) and Jeannin and Pistorius (2008), approximating Lévy processes by HEJD processes is the central point of this dissertation. Our starting point is the desire to find a more systematic methodology of approximating Lévy processes by HEJD processes which has the following six features:

1. No non-linear least-squares fitting is required.
2. No guessing of the mean jump sizes is required.
3. The methodology is equally as intuitive and easy to implement as the procedures just described above.
4. The methodology has a robust way of approximating the very small jumps by Brownian motion.
5. The methodology yields more accurate vanilla option prices than the procedures just described above.
6. The methodology yields more accurate barrier option prices than the procedures just described above.

Three comments are in order about the last six points. Firstly, our desire to avoid non-linear least-squares fitting is based on the fact that such methods can be unstable or ill-posed because the algorithm may find a local minimum rather than the global minimum. Secondly, although we wish to have more accurate vanilla option prices, this will primarily be in order that we can benchmark the accuracy of our methodology (since we know we can accurately price vanilla options for both General Classes of Lévy processes and for the HEJD process (see Lipton (2001), Lewis (2001), Sepp (2003) or section 5). Thirdly, although we will focus mostly on pricing barrier options, we believe our methodology should also be of interest for pricing other exotic options, either by Laplace transform methods (for example, for lookback options and Russian options, see Kou and Wang (2004) and Sepp (2005)) or by Monte Carlo simulation.

We describe our methodology, which does in fact have our six desirable features, in this dissertation. The rest of this dissertation is structured as follows:

In section 2 we give a general overview of the Lévy processes we will use.
In section 3 we review some techniques for simulating Lévy processes. In section 4 we explain the algorithm that calculates the parameters of the approximating hyperexponential process.
In sections 5 and 6 we present our results for vanilla option prices and for barrier option prices.
In section 7 we give the conclusions of this dissertation. In the appendices, we give some details about some of the methodologies we use and we present our results in tabular and graphical forms.

2 Lévy Processes

2.1 Definition

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space, and $\{\mathcal{F}_t\}_{t_0 \leq t < \infty}$ a filtration which we assume satisfies the usual conditions, with $t_0 \equiv 0$.

Definition 2.1.1. A Lévy process X_t is a càdlàg stochastic process with $X_{t_0} = 0$ with probability one, that satisfies the following conditions:

- X_t has independent increments, i.e. for any $t_0 \leq s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.
- X_t has stationary increments, i.e. for $t_0 \leq t$, $0 \leq s$, the distribution of $X_{t+s} - X_t$ does not depend on t.
- X_t is stochastically continuous, i.e. for any $\epsilon > 0$, $t_0 \leq s$, $\lim_{t \to s} P(|X_t - X_s| > \epsilon) = 0$.

Moreover, if $\Phi_t(u) = \mathbb{E}[\exp(iuX_t)]$ is the characteristic function of X_t, then it is infinitely divisible, i.e. for any $n \in \mathbb{N}$, there exists a characteristic function $\phi_{t,n}(u)$ such that for each $u \in \mathbb{R}$, $\Phi_t(u) = (\Phi_{t,n}(u))^n$. The characteristic exponent $\phi(u)$, defined by $\Phi_t(u) = \exp((t - t_0)\phi(u))$ satisfies the Lévy-Khintchine formula.

Theorem 2.1.2. (Lévy-Khintchine formula)
If X_t is a Lévy process, then $\Phi_t(u) = \exp((t - t_0)\phi(u))$, for each $t \geq t_0$, $u \in \mathbb{R}$, where

$$\phi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux\mathbf{1}_{\{|x|<1\}})\nu(dx), \tag{2.1}$$

for some $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and a measure ν on $\mathbb{R}\backslash\{0\}$ with $\int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(dx) < \infty$.

The characteristics $(\gamma, \sigma^2, \nu(dx))$ are called the Lévy triplet. ν is called the Lévy measure. If $\nu(dx) = \nu(x)\, dx$, then we also call $\nu(x)$ the Lévy density.

Proposition 2.1.3. *Let X_t be a Lévy process with triplet (γ, σ^2, ν)*

- *If $\nu(\mathbb{R}) < \infty$, then almost all paths of X_t have a finite number of jumps on every compact interval. The Lévy process has finite activity.*
- *If $\nu(\mathbb{R}) = \infty$, then almost all paths of X_t have an infinite number of jumps on every compact interval. The Lévy process has infinite activity.*

Proposition 2.1.4. *Let X_t be a Lévy process with triplet (γ, σ^2, ν)*

- *If $\sigma^2 = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, then almost all paths of X_t have finite variation.*
- *If $\sigma^2 \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, then almost all paths of X_t have infinite variation.*

Remark 2.1.5. We need the term $iux\mathbf{1}_{\{|x|<1\}}$ in equation (2.1) in general so that the integral converges. However, if the Lévy density decays fast enough as $x \to \pm\infty$, (which is the case for the Lévy processes we will consider here, such as CGMY and NIG, but not the case in general, for the α-stable process for example), we can replace $iux\mathbf{1}_{\{|x|<1\}}$ by iux. If the process is of finite variation, then we do not need this term at all. We remark that changing the form of this term is essentially the same as changing the drift of X_t.

We define the mean-corrected characteristic exponent $\phi_{MC}(u)$, via $\phi_{MC}(u) = \phi(u) - iu\phi(-i)$. It is straightforward to see that $\phi_{MC}(u)$ is the characteristic exponent of a Lévy process X_t which satisfies $\mathbb{E}\left[e^{X_t}\right] = 1$. Since in applications to option pricing theory the drift of the Lévy process is determined by considerations of no arbitrage (see equation (1.2)), it is sufficient to consider Lévy processes which satisfy $\mathbb{E}\left[e^{X_t}\right] = 1$. Since we shall only consider Lévy processes whose Lévy density decays fast enough as $x \to \pm\infty$, we can always write the characteristic exponent in the form:

$$\begin{aligned}\phi(u) \;=\; & -\frac{1}{2}\sigma^2(u^2 + i\alpha u) + \int_{-\infty}^{+\infty}(e^{iux} - 1 - i\beta zx)\nu(x)dx \\ & - i\alpha u\int_{-\infty}^{+\infty}(e^x - 1 - \beta x)\nu(x)dx,\end{aligned}$$

where $\alpha = 1$ if we decide to use the mean-corrected characteristic exponent, $\alpha = 0$ otherwise, and $\beta = 1$ for processes of infinite variation, and $\beta = 0$ otherwise.

In this dissertation, we only consider Lévy processes whose Lévy density is defined on the whole of the real line. We can therefore write:

$$\nu(x) \equiv \nu_+(x)\mathbf{1}_{\{x>0\}} + \nu_-(-x)\mathbf{1}_{\{x<0\}}.$$

If the terms $\nu_+(x)$ and $\nu_-(-x)$ are completely monotonic (and for all the Lévy processes we will consider, $\nu_+(x)$ and $\nu_-(-x)$ are easily verified to be completely monotonic), then by Bernstein's theorem we can write $\nu(x)$ in the form:

$$\nu(x) = \mathbf{1}_{\{x>0\}}\int_0^{+\infty} e^{-ux}\mu_+(u)\;du + \mathbf{1}_{\{x<0\}}\int_{-\infty}^{0} e^{-|ux|}\mu_-(u)\;du, \tag{2.2}$$

where $\mu_+(u)$ and $\mu_-(u)$ are some measures on $(0,\infty)$ and $(-\infty,0)$ respectively, i.e. $\mu_+(u)$ and $\mu_-(u)$ are non-negative.

2.2 The Normal Inverse Gaussian Process

The Normal Inverse Gaussian distribution has been introduced by Eberlein and Keller (1995) and Barndorff-Nielsen (1995), as a subclass of the generalized hyperbolic distributions. It has parameters $\alpha > 0$,$-\alpha < \beta < \alpha$ and $\delta > 0$. NIG(α,β,δ) characteristic function is given by

$$\Phi_{NIG}(u;\alpha,\beta,\delta) = exp(-\delta(\sqrt{\alpha^2 - (\beta+iu)^2} - \sqrt{\alpha^2-\beta^2})).$$

The NIG process X_t is a Lévy process defined with $X_{t_0} = 0$ and stationary and independent increments distributed according to the NIG distribution. Its Lévy measure is given by

$$\nu_{NIG}(x) = \frac{\delta\alpha}{\pi}\frac{exp(\beta x)K_1(\alpha|x|)}{|x|},$$

where $K_1(x)$ is the modified Bessel function of the second kind. We will use the following representation of K_1

$$K_1(x) = x\int_1^{+\infty} e^{-xv}\sqrt{v^2-1}dv,$$

The NIG process has infinite activity and infinite variation. We obtain for the measures μ_+ and μ_-

$$\mu_+(u) = \frac{\delta\alpha}{\pi}\sqrt{\left(\frac{u+\beta}{\alpha}\right)^2 - 1}\ \mathbf{1}_{\{u>\alpha-\beta\}}$$

and

$$\mu_-(u) = \frac{\delta\alpha}{\pi}\sqrt{\left(\frac{u+\beta}{\alpha}\right)^2 - 1}\ \mathbf{1}_{\{-u>\alpha+\beta\}}.$$

The characteristics of the NIG distribution are (Schoutens (2003)):

	$NIG(\alpha, \beta, \delta)$
mean	$\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}$
variance	$\frac{\alpha^2\delta}{(\alpha^2-\beta^2)^{3/2}}$
skewness	$\frac{3\beta}{\alpha\delta^{1/2}(\alpha^2-\beta^2)^{1/4}}$
kurtosis	$3\left(1+\frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}\right)$

2.3 The CGMY Process

The CGMY process was introduced by Carr, Geman, Madan and Yorr (2002), and is also called the KoBoL process. For $Y \neq 0, 1, 2$, its characteristic function is given by

$$\Phi_{CGMY}(u; C, G, M, Y) = \exp(C\Gamma(-Y)\left[(M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right]),$$

where $C, G, M > 0$ and $Y < 2$.
The Lévy measure of the CGMY process is

$$\nu_{CGMY}(x) = C\frac{e^{-Mx}}{x^{1+Y}}\mathbf{1}_{\{x>0\}} + C\frac{e^{-G|x|}}{|x|^{1+Y}}\mathbf{1}_{\{x<0\}}.$$

The parameter Y determines the character of both the activity and the variation of the CGMY process.

- For $Y < 0$, the CGMY process has finite activity and finite variation,
- for $0 \leq Y < 1$, the process has infinite activity and finite variation
- and for $1 \leq Y < 2$, it has infinite activity and infinite variation.

The measures μ_+ and μ_- are obtained using the following formula:

$$\frac{1}{x^{1+Y}} = \int_0^{+\infty} e^{-ux} \frac{u^Y}{\Gamma(1+Y)} du,$$

and they are given by

$$\mu_+(u) = C\frac{(u-M)^Y}{\Gamma(1+Y)} \mathbf{1}_{\{u>M\}}$$

and

$$\mu_-(u) = C\frac{(-u-G)^Y}{\Gamma(1+Y)} \mathbf{1}_{\{-u>G\}}.$$

The characteristics of the CGMY distribution are (Schoutens (2003)):

	$CGMY(C,G,M,Y)$
mean	$C(M^{Y-1} - G^{Y-1})\Gamma(1-Y)$
variance	$C(M^{Y-2} + G^{Y-2})\Gamma(2-Y)$
skewness	$\dfrac{C(M^{Y-3} - G^{Y-3})\Gamma(3-Y)}{(C(M^{Y-2} + G^{Y-2})\Gamma(2-Y))^{3/2}}$
kurtosis	$3 + \dfrac{C(M^{Y-4} + G^{Y-4})\Gamma(4-Y)}{(C(M^{Y-2} + G^{Y-2})\Gamma(2-Y))^2}$

2.4 The Variance Gamma Process

The Variance Gamma process can be obtained by setting Y = 0 in the CGMY model. It has two sets of parameterization: in terms of C,G and M, and in terms of σ, ν, θ, where

$$\begin{aligned} C &= 1/\nu, \\ G &= (\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu)^{-1}, \\ M &= (\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu)^{-1}. \end{aligned}$$

The characteristic function of the VG law is given by

$$\Phi_{VG}(u; \sigma, \nu, \theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-1/\nu},$$

or, with the C,G,M parameterization

$$\Phi_{VG}(u; C, G, M) = \left(\frac{GM}{GM + (M-G)iu + u^2}\right)^C.$$

The Lévy measure for the VG process is given by

$$\nu(x) = C\frac{e^{-Mx}}{x}\mathbf{1}_{\{x>0\}} + C\frac{e^{-G|x|}}{|x|}\mathbf{1}_{\{x<0\}}.$$

The VG process has infinite activity and finite variation. The measures μ_+ and μ_- are obtained by setting Y = 0 in the CGMY model

$$\mu_+(u) = C\mathbf{1}_{\{u>M\}}$$

and

$$\mu_-(u) = C\mathbf{1}_{\{-u>G\}}.$$

The characteristics of the VG distribution are, with the (σ, ν, θ) parameterization (Schoutens (2003)):

	$VG(\sigma, \nu, \theta)$
mean	θ
variance	$\sigma^2 + \nu\theta$
skewness	$\theta\nu\frac{(3\sigma^2 + 2\nu\theta^2)}{(\sigma^2 + \nu\theta^2)^{3/2}}$
kurtosis	$3(1 + 2\nu - \frac{\nu\sigma^4}{(\sigma^2 + \nu\theta^2)^2})$

and with the CGM parameterization:

	$VG(C, G, M)$
mean	$C\left(\frac{1}{M} - \frac{1}{G}\right)$
variance	$C\left(\frac{1}{M^2} + \frac{1}{G^2}\right)$
skewness	$2C^{-1/2}\frac{(G^3 - M^3)}{(G^2 + M^2)^{3/2}}$
kurtosis	$3\left(1 + 2C^{-1}\frac{(G^4 + M^4)}{(G^2 + M^2)^2}\right)$

2.5 The Hyperexponential Jump-Diffusion Process

The hyperexponential jump-diffusion process (henceforth HEJD), with an arbitrary number N (we assume N is even for notational simplicity) of compound Poisson processes has the form

$$X_t = \sigma W_t + \sum_{i=1}^{N/2} \rho_i \sum_{k=1}^{N_{t,i}} J_i + \sum_{i=N/2+1}^{N} \rho_i \sum_{k=1}^{N_{t,i}} J_i,$$

where $N_{t,i}$, for each $i = 1, 2, ..., N$, denotes a Poisson (counting) process with $N_{t_0,i} = 0$, J_i, for each $i = 1, 2, ..., N$, are independent random variables which are exponentially distributed, $\rho_i = 1$, if $1 \leq i \leq \frac{N}{2}$ (the corresponding jumps are up jumps) and $\rho_i = -1$, if $\frac{N}{2} + 1 \leq i \leq N$ (the corresponding jumps are down jumps).
Furthermore, we denote by a_i and c_i respectively the intensity rates of the Poisson processes

corresponding to up jumps and down jumps, and we denote by b_i and d_i the reciprocals of the mean jump sizes for the up and down jumps respectively, i.e.

$$\begin{aligned}\mathbb{E}[N_{t,i}] &= a_i t,\ \mathbb{E}[J_i] = \frac{1}{b_i}, \quad \text{for } 1 \le i \le \frac{N}{2}, \\ \mathbb{E}[N_{t,i}] &= c_i t,\ \mathbb{E}[J_i] = \frac{1}{d_i}, \quad \text{for } \frac{N}{2}+1 \le i \le N.\end{aligned}$$

The Lévy measure is given by

$$\nu(x) = \sum_{i=1}^{N/2} a_i b_i e^{-b_i x}\mathbf{1}_{\{x>0\}} + \sum_{i=N/2+1}^{N} c_i d_i e^{d_i x}\mathbf{1}_{\{x<0\}}.$$

The measures μ_+ and μ_- are

$$\mu_+(u) = \sum_{i=1}^{N/2} a_i b_i \delta_{b_i}(u) \text{ and } \mu_-(u) = \sum_{i=N/2+1}^{N} c_i d_i \delta_{-d_i}(u),$$

where δ denotes the Dirac delta function.
The characteristic function of this process has the form

$$\begin{aligned}\mathbb{E}[e^{iuX_t}] &= \mathbb{E}\left[\exp\left(iu\left(\sigma W_t + \sum_{i=1}^{N/2}\rho_i\sum_{k=1}^{N_{t,i}} J_i + \sum_{i=N/2+1}^{N}\rho_i\sum_{k=1}^{N_{t,i}} J_i\right)\right)\right] \\ &= \exp\left(-\frac{1}{2}u^2\sigma^2 t + t\left[\sum_{i=1}^{N/2} a_i\mathbb{E}[e^{iuJ_i}-1] + \sum_{i=N/2+1}^{N} c_i\mathbb{E}[e^{iuJ_i}-1]\right]\right) \\ &= \exp\left(-\frac{1}{2}u^2\sigma^2 t + t\left[\sum_{i=1}^{N/2} a_i b_i\left(\frac{1}{b_i - iu} - \frac{1}{b_i}\right)\right.\right. \\ &\qquad \left.\left. + \sum_{i=N/2+1}^{N} c_i d_i\left(\frac{1}{d_i+iu} - \frac{1}{d_i}\right)\right]\right).\end{aligned}$$

The characteristics of the HEJD distributions are:

	$HEJD(a,b,c,d,\sigma)$
mean	$\sum_{i=1}^{N/2}\frac{a_i}{b_i} - \sum_{i=N/2+1}^{N}\frac{c_i}{d_i}$
variance	$\sigma^2 + 2\left(\sum_{i=1}^{N/2}\frac{a_i}{b_i^2} + \sum_{i=N/2+1}^{N}\frac{c_i}{d_i^2}\right)$
skewness	$\frac{6\left(\sum_{i=1}^{N/2} a_i/b_i^3 - \sum_{i=N/2+1}^{N} c_i/d_i^3\right)}{(\sigma^2 + 2\left(\sum_{i=1}^{N/2} a_i/b_i^2 + \sum_{i=N/2+1}^{N} c_i/d_i^2\right))^{3/2}}$
kurtosis	$3\left(1 + 8\frac{\sum_{i=1}^{N/2} a_i/b_i^4 + \sum_{i=N/2+1}^{N} c_i/d_i^4)}{\left(\sigma^2 + 2\left(\sum_{i=1}^{N/2} a_i/b_i^2 + \sum_{i=N/2+1}^{N} c_i/d_i^2\right)\right)^2}\right)$

3 Monte Carlo simulation

There are a number of methods for simulating Lévy processes and we will review some of them briefly in this section. The review is not exhaustive and we refer the reader to Schoutens (2003), Cont and Tankov (2004) and Glasserman (2004) for more detailed information. Most of the methods essentially approximate a Lévy process by a jump-diffusion process. We will only review these methods and we do so in the hope that they may give us some intuition as to how best to approximate a Lévy process by a HEJD process. To simplify our brief exposition and to allow us to focus on the intuition as opposed to the details of the mathematics, we will only consider how to simulate a CGMY process when the parameter Y satisfies $0 \leq Y < 1$. This means that the process has infinite activity and finite variation. We write the process X_t in the form: $X_t = X_t^+ - X_t^-$ where X_t^+ and X_t^- are independent spectrally positive Lévy processes (ie they only have up jumps - the down jumps present in CGMY come from the minus sign). We choose ϵ_+ and ϵ_- such that $0 < \epsilon_+,\ \epsilon_- << 1$.
From the Lévy-Ito decomposition, we know we can represent the process X_t in the form:

$$X_t = X_t^+ - X_t^- = \gamma t + X_t^+(\epsilon_+) - X_t^-(\epsilon_-) + R_t^+(\epsilon_+) - R_t^-(\epsilon_-),$$

where γ is a drift term which will eventually be immaterial for pricing options because the drift is specified by no-arbitrage arguments (see equation (1.2)), $X_t^+(\epsilon_+)$ and $X_t^-(\epsilon_-)$ are independent compound Poisson processes with intensity rates given by
$C\int_{\epsilon_+}^{\infty}(\exp(-Mx)/x^{Y+1})\,dx$ and $C\int_{\epsilon_-}^{\infty}(\exp(-Gx)/x^{Y+1})\,dx$ respectively and the residual terms $R_t^+(\epsilon_+)$ and $R_t^-(\epsilon_-)$ are independent Lévy processes with finite mean and variance (this follows since the jumps are bounded) and Lévy densities of the form $I(x \leq \epsilon_+)\exp(-Mx)/x^{Y+1}$ and $I(x \leq \epsilon_-)\exp(-Gx)/x^{Y+1}$ respectively.

We now consider seperately how we can approximate, firstly, $R_t^+(\epsilon_+)$ and $R_t^-(\epsilon_-)$ and then, secondly $X_t^+(\epsilon_+)$ and $X_t^-(\epsilon_-)$.

3.1 Approximation of the residual terms $R_t^+(\epsilon_+)$ and $R_t^-(\epsilon_-)$

The simplest approximation for $R_t^+(\epsilon_+)$ and $R_t^-(\epsilon_-)$ would be to replace them by their expected values (the actual expected values are immaterial for pricing options as already indicated). This is equivalent to making them deterministic and hence a measure of the level of the approximation involved is to consider the variance of these terms (which we would like to be very small). For small ϵ_+ and ϵ_-, the variance is:

$$Var(R_t^+(\epsilon_+) - R_t^-(\epsilon_-)) \approx t(\int_0^{\epsilon_+} x^2/x^{Y+1}\,dx + \int_0^{\epsilon_-} x^2/x^{Y+1}\,dx) = t(\frac{\epsilon_+^{2-Y}}{1-Y} + \frac{\epsilon_-^{2-Y}}{1-Y})$$

. When Y is close to zero, the approximation will be better and conversely when Y is close to one, the approximation will be very bad.

The next level of approximation for $R_t^+(\epsilon_+)$ and $R_t^-(\epsilon_-)$ would be to replace them by Brownian motion with variance term $\frac{\epsilon_+^{2-Y}}{1-Y} + \frac{\epsilon_-^{2-Y}}{1-Y}$.
Asmussen and Rosinski (2001) show that this is justified when $\frac{\epsilon_+^{2-Y}}{(1-Y)\epsilon_+^2} + \frac{\epsilon_-^{2-Y}}{(1-Y)\epsilon_-^2}$ tends to ∞ as ϵ_+ and ϵ_- tend to zero.

This last expression holds true when the Y parameter is strictly positive but not when $Y = 0$ (which corresponds to VG). If Y is close to one, the approximation by Brownian motion is very accurate whereas if Y is close to zero, the approximation is relatively poor.

This is intuitive since the larger the value of Y, the more small jumps there are. A Central Limit Theorem type of argument then suggests that the approximation by Brownian motion is better. We will assume in the next section that we have approximated $R_t^+(\epsilon_+)$ and $R_t^-(\epsilon_-)$ by Brownian motion.

3.2 Approximation of $X_t^+(\epsilon_+)$ and $X_t^-(\epsilon_-)$

We now consider how we might approximate the independent compound Poisson processes $X_t^+(\epsilon_+)$ and $X_t^-(\epsilon_-)$.
One methodology (see p103-106, Schoutens (2003)) consists of discretizing the Lévy density $\nu(x)$ of the CGMY process.
We make a partition of the form $-\infty < a_0 < a_1 < ... < a_k = -\epsilon_-$ and $\epsilon_+ = a_{k+1} < a_{k+2} < ... < a_{d+1} < \infty$. We construct independent Poisson processes $N_t^{-,i}$ and $N_t^{+,j}$ for each i, $i = 1, ..., k$ and j, $j = k+1, ..., d$ with intensity rates respectively given by $\int_{a_{i-1}}^{a_i} \nu(x)\,dx$ and $\int_{a_j}^{a_{j+1}} \nu(x)\,dx$ respectively.
In Schoutens (2003), the amplitudes of the jumps are fixed and are chosen so that the variance of the Poisson process matches the part of the variance of the Lévy process corresponding to this interval. However, other schemes are possible. Although Schoutens (2003) does suggest three possible ways that the parameters $a_0 < a_1 < ... < a_k$ and $a_{k+1} < a_{k+2} < ... < a_{d+1}$ could be chosen, none of them seems very concrete or, in any sense, optimal. As such, this is a little analogous to guessing the mean jump sizes when approximating a Lévy process by a HEJD process.

A much more useful and concrete methodology is described on p108 of Schoutens (2003) and in section 6.5 of Cont and Tankov (2004).

The result relies on a series representation which can be applied to the independent compound Poisson processes $X_t^+(\epsilon_+)$ and $X_t^-(\epsilon_-)$ seperately.

Proposition 3.2.1. *Let $(Z_t)_{t\geq 0}$ be a subordinator (ie a non-negative, non-decreasing Lévy process) whose Lévy density $\nu(x)$ has tail integral $U(x) = \int_x^\infty \nu(\chi)\,d\chi$.
Let Γ_i be a sequence of jumping times of a standard (ie unit mean) Poisson process, and V_i be an independent sequence of independent random variables, uniformly distributed on $[0,1]$. Then $(Z_t)_{t\geq 0}$ is representable in law, on the time interval $[0,1]$ as:*

$$\{(Z_s), 0 \leq s \leq 1\} \equiv \{(\hat{Z}_s), 0 \leq s \leq 1\}, \text{ in law,}$$

with $\hat{Z}_s = \sum_{i=1}^{\infty} U^{-1}(\Gamma_i) I_{V_i \leq s}$, where the generalised inverse $U^{-1}(y)$ is defined by $U^{-1}(y) = inf\{x > 0 : U(x) < y\}$.

Furthermore, if we fix some $\epsilon > 0$ and let $N(\epsilon) = inf\{: U^{-1}(\Gamma_i) \geq \epsilon\}$, then the trucated series $\hat{Z}_s^\epsilon = \sum_{i=1}^{N(\epsilon)} U^{-1}(\Gamma_i) I_{V_i \leq s}$ is a compound Poisson process with Lévy measure defined on $[\epsilon, \infty)$.

Proof: See p192-194 of Cont and Tankov (2004).

Note that the right way to truncate the infinite series is not to keep a fixed number of terms but to fix ϵ. This means that $N(\epsilon)$ will be random.

The above series representation can be used for $X_t^+(\epsilon_+)$ and $X_t^-(\epsilon_-)$ seperately. The form of the inverse tail mass function $U^{-1}(y)$ is known (Gander and Stephens (2004)) and is given by

$$U^{-1}(y) = (\frac{C}{y})^{1/Y} \exp(-L_W((\frac{M}{Y})(\frac{C}{y})^{1/Y}))$$

where $L_W(x)$ is the Lambert-W function defined via $x = L_W(x)\exp(L_W(x))$. This is a function which can be evaluated very rapidly to machine tolerance and is therefore, for practical purposes, analytic. The form of $U^{-1}(y)$ above is for $X_t^+(\epsilon_+)$. The form for $X_t^-(\epsilon_-)$ simply replaces M by G.

The question that now needs to be adressed is how to choose ϵ. In principle, we would like ϵ to be close to machine tolerance (approximately 10^{-14} or 10^{-15}). With this choice, there would, for practical purposes, be no error in the sense that the Brownian component would be so small as to be negligible and the compound Poisson approximation to the CGMY process would be almost perfect.
On the other hand, we don't want the mean number of terms in the series representation to be too large - otherwise, computation of the series representation will be too slow for practical purposes. The problem is that the mean number of terms is very sensitive to the value of the Y parameter and also quite sensitive to the value of ϵ. The following data is taken from Crosby (2007).

ϵ	1.e-08	1.e-11	1.e-14
Y	Mean number of terms		
0.121	3	7	17
0.321	59	498	4930
0.521	548	24077	N/A
0.721	5334	481480	N/A
0.921	32521	N/A	N/A

It shows the mean number of terms required over 1000 Monte carlo simulations for different values of Y and ϵ. The C, M and G parameters were chosen as typical values obtained when the CGMY process is calibrated to the market prices of vanilla fx options. N/A means that the mean number of terms is so large that it was not possible to compute it in the computer time that was available.

For small values of Y, eg $Y \approx 0.15$, the mean number of terms is very small eg less than twenty even for a value of ϵ of 10^{-14}. This means that it actually is practical to choose a value of ϵ close to machine tolerance.
However as Y increases, the mean number of terms grows extremely rapidly. The data shows that if $Y \approx 0.5$, even a choice of ϵ of 10^{-8} may not be practical. However, we know from section 3.1, that as Y increases, the quality of the approximation of the small jumps by Brownian motion improves which means a larger value of ϵ could be justified from that point of view.
On the other hand, a larger value of ϵ means that the compound Poisson approximation is less good. When Y is small, we can choose ϵ to be so small that the compound Poisson approximation is almost perfect. However, then the quality of the approximation of the small jumps by Brownian motion is relatively poor.

3.3 Intuition and discussion

The purpose of this dissertation is to approximate a Lévy process by a HEJD process. It is worth pausing to reflect upon what intuition the above brief discussion on methods of simulating Lévy processes may give us.

We might expect that for a CGMY process, keeping C, M and G constant, larger values of Y will lead to larger diffusion volatilities. Conversely, when Y is zero (ie we have a VG process), we would expect the diffusion volatility to be very small (if not zero). Taking the intuition of the Central Limit Theorem a step further, we might expect infinite variation processes (eg CGMY with $Y \geq 1$ or NIG) to tend to have larger diffusion volatilties than finite variation processes (clearly the actual values will depend upon the parameters of the Lévy process, so we concede this statement is not precise). Results which we report in section 5 and in the appendices largely support this intuition.

What the above discussion does not tell us is what are appropriate values for ϵ, ϵ_+ and ϵ_-.

When using the series representation (proposition 3.2.1) for Monte Carlo simulation, we fix ϵ and the number of terms used in the series is random. When approximating a Lévy process by a HEJD process, we will be given the number of Poisson processes to be used in the approximation (the number of Poisson processes will be one of the main determinants of calculation times and so it seems sensible to choose that number at the outset). If we translate the intuition about the series representation with a random number of terms above into intuition about approximating a Lévy process by a HEJD process with a fixed number of Poisson processes, we might expect the following:

For a given value of Y, if we were to have a smaller diffusion volatility, because we effectively truncate the small jumps at a smaller value (in order to approximate them by Brownian motion), then the approximation of the small jumps by Brownian motion becomes more accurate but the approximation of the larger jumps of the Lévy process by compound Poisson processes with double exponentially distributed jumps becomes less accurate.

Conversely, for the same value of Y, if we were to have a larger diffusion volatility, because we effectively truncate the small jumps at a larger value (in order to approximate them by Brownian motion), then the approximation of the small jumps by Brownian motion becomes less accurate but the approximation of the larger jumps of the Lévy process by compound Poisson processes with double exponentially distributed jumps becomes more accurate.

Hence, there is a subtle trade-off between the two different approximations.

4 Model

4.1 Presentation

Our aim in this section is to approximate, in distribution, a given Lévy process by a HEJD process. We have already remarked in the introduction that equation (1.3) immediately suggests such an approximation.

Asmussen et al. (2007) and Jeannin and Pistorius (2008) use this as the basis of their approximation scheme.

We know that we can demonstrate convergence in distribution if we can demonstrate convergence in characteristic exponent (or characteristic function). Hence, we seek to approximate the characteristic exponent of the Lévy process by the characteristic exponent of a HEJD process. We will assume that the Lévy process has no gaussian component and its Lévy density decays fast enough as $x \to \pm\infty$ (see section 2.1) to justify remark 2.1.5.

From Schoutens (2003), Cont and Tankov (2004), or section 2.1, we can write the characteristic exponent $\phi(z),\ z \in \mathbb{C}$, of the Lévy process in the form:

$$\begin{aligned}\phi(z) &= \int_{-\infty}^{+\infty} (e^{izx} - 1 - i\beta zx)\nu(x)\ dx \\ &- i\alpha z \int_{-\infty}^{+\infty} (e^{x} - 1 - \beta x)\nu(x)\ dx. \qquad (4.1)\end{aligned}$$

Given N, we seek to approximate this by the characteristic exponent $\phi_N(z)$ of a HEJD process:

$$\begin{aligned}\phi_N(z) &= \sum_{i=1}^{N/2} a_i b_i \left(\frac{1}{b_i - iz} - \frac{1}{b_i}\right) - i\alpha z \sum_{i=1}^{N/2} a_i b_i \left(\frac{1}{b_i - 1} - \frac{1}{b_i}\right) \\ &+ \sum_{i=N/2+1}^{N} c_i d_i \left(\frac{1}{d_i + iz} - \frac{1}{d_i}\right) - i\alpha z \sum_{i=1+N/2}^{N} c_i d_i \left(\frac{1}{d_i + 1} - \frac{1}{d_i}\right) \\ &- \frac{1}{2}\sigma^2(z^2 + i\alpha z)\end{aligned}$$

where we must choose $a_i,\ b_i,\ c_i,\ d_i$ and σ^2, given N, so as to make the approximation as accurate as possible. Note that we use the mean-corrected characteristic exponent by choosing $\alpha = 1$.

If we substitute equation (2.2) into the equation (4.1), we get

$$\begin{aligned}\phi(z) &= \int_0^{+\infty}\int_0^{+\infty} e^{-ux}\mu_+(u)(e^{izx} - 1 - i\beta zx)\ du\ dx \\ &- i\alpha z \int_0^{+\infty}\int_0^{+\infty} e^{-ux}\mu_+(u)(e^{x} - 1 - \beta x)\ du\ dx \\ &+ \int_{-\infty}^{0}\int_{-\infty}^{0} e^{-|ux|}\mu_-(u)(e^{izx} - 1 - i\beta zx)\ du\ dx \\ &- i\alpha z \int_{-\infty}^{0}\int_{-\infty}^{0} e^{-|ux|}\mu_-(u)(e^{x} - 1 - \beta x)\ du\ dx.\end{aligned}$$

By changing variables ($u \to -u$ and $x \to -x$ in the $\int_{-\infty}^{0}$ integrals) and switching the order of the integration, we get

$$\begin{aligned}
\phi(z) &= \int_0^{+\infty}\int_0^{+\infty} e^{-ux}\mu_+(u)(e^{izx}-1-i\beta zx)\;dx\;du \\
&- i\alpha z\int_0^{+\infty}\int_0^{+\infty} e^{-ux}\mu_+(u)(e^{x}-1-\beta x)\;dx\;du \\
&+ \int_0^{+\infty}\int_0^{+\infty} e^{-ux}\mu_-(-u)(e^{-izx}-1+i\beta zx)\;dx\;du \\
&- i\alpha z\int_0^{+\infty}\int_0^{+\infty} e^{-ux}\mu_-(-u)(e^{-x}-1+\beta x)\;dx\;du \\
\phi(z) &= \int_0^{+\infty}\mu_+(u)\left(\frac{1}{u-iz}-\frac{1}{u}-\frac{i\beta z}{u^2}\right)\;du \\
&- i\alpha z\int_0^{+\infty}\mu_+(u)\left(\frac{1}{u-1}-\frac{1}{u}-\frac{\beta}{u^2}\right)\;du \\
&+ \int_0^{+\infty}\mu_-(-u)\left(\frac{1}{u+iz}-\frac{1}{u}+\frac{i\beta z}{u^2}\right)\;du \\
&- i\alpha z\int_0^{+\infty}\mu_-(-u)\left(\frac{1}{u+1}-\frac{1}{u}+\frac{\beta}{u^2}\right)\;du,
\end{aligned}$$

that we can rewrite as:

$$\phi(z) = \int_0^{+\infty}\mu_+(u)g^+_{\alpha,\beta}(u,z)\;du \;+\; \int_0^{+\infty}\mu_-(-u)g^-_{\alpha,\beta}(u,z)\;du,$$

where

$$\begin{aligned}
g^+_{\alpha,\beta}(u,z) &\equiv \left(\frac{1}{u-iz}-\frac{1}{u}-\frac{i\beta z}{u^2}\right)-i\alpha z\left(\frac{1}{u-1}-\frac{1}{u}-\frac{\beta}{u^2}\right) \\
&= \frac{iz(u+iz)\left[u(u-1)-(u-iz)\left(\beta(1-\alpha)(u-1)+\alpha u\right)\right]}{u^2(u-1)(u^2+z^2)}
\end{aligned}$$

and

$$\begin{aligned}
g^-_{\alpha,\beta}(u,z) &\equiv \left(\frac{1}{u+iz}-\frac{1}{u}+\frac{i\beta z}{u^2}\right)-i\alpha z\left(\frac{1}{u+1}-\frac{1}{u}+\frac{\beta}{u^2}\right) \\
&= \frac{iz(u-iz)\left[-u(u+1)-(u+iz)\left(\beta(\alpha-1)(u+1)-\alpha u\right)\right]}{u^2(u+1)(u^2+z^2)}.
\end{aligned}$$

We recognize that the terms $g^+_{\alpha,\beta}(u,z)$ and $g^-_{\alpha,\beta}$ are the terms appearing in the characteristic exponents, mean-corrected or not, of exponentially distributed processes, for up jumps and for down jumps respectively,with mean jump sizes $\frac{1}{u}$ and intensity rates $\frac{1}{u}$. In the case where $\beta = 1$, the terms $-\frac{iz}{u^2}$ and $\frac{iz}{u^2}$ are simply additional (and, from an option pricing perspective, irrelevant) drift terms, which in any event cancel out if $\alpha = 1$.

Now we define

$$h^+_{\alpha,\beta}(u,z) \equiv -\frac{g^+_{\alpha,\beta}(u,z)}{z^2+i\alpha z} \quad\text{and}\quad h^-_{\alpha,\beta}(u,z) \equiv -\frac{g^-_{\alpha,\beta}(u,z)}{z^2+i\alpha z}.$$

Then if we introduce $\overline{\omega}_+,\ \overline{\omega}_- \ \epsilon\ \mathbb{R}^+$ and split the integrals in two, we can write the characteristic function as

$$\begin{aligned}\phi(z) &= \int_0^{\overline{\omega}_+} \mu_+(u) g^+_{\alpha,\beta}(u,z)\ du\ +\ \int_0^{\overline{\omega}_-} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\ du \\ &- (z^2 + i\alpha z)\int_{\overline{\omega}_+}^{+\infty} \mu_+(u) h^+_{\alpha,\beta}(u,z)\ du \\ &- (z^2 + i\alpha z)\int_{\overline{\omega}_-}^{+\infty} \mu_-(-u) h^-_{\alpha,\beta}(u,z)\ du. \end{aligned} \tag{4.2}$$

The form of this last equation immediately suggest an approximation scheme. Replacing the first two integrals by summations, it suggests the following approximation:

$$\begin{aligned}\phi(z) &\simeq \sum_{i=1}^{N/2} \omega_i^+ \mu_+(u_i^+) g^+_{\alpha,\beta}(u_i^+, z) \quad + \sum_{i=1+N/2}^{N} \omega_i^- \mu_-(-u_i^-) g^-_{\alpha,\beta}(u_i^-, z) \\ &- \frac{1}{2}(\Sigma^{(+)2} + \Sigma^{(-)2})(z^2 + i\alpha z), \end{aligned} \tag{4.3}$$

where ω_i^+ and u_i^+, for $i = 1, ..., N/2$, are respectively weights and abscissas coming from a $N/2$-point Gaussian quadrature rule on the interval $(0, \overline{\omega}_+)$ and where ω_i^- and u_i^-, for $i = 1{+}N/2, ..., N$, are respectively weights and abscissas coming from a $N/2$-point Gaussian quadrature rule on the interval $(0, \overline{\omega}_-)$, and where

$$\frac{1}{2}\Sigma^{(+)2} = \int_{\overline{\omega}_+}^{+\infty} \mu_+(u) h^+_{\alpha,\beta}(u,z) du$$

and

$$\frac{1}{2}\Sigma^{(-)2} = \int_{\overline{\omega}_-}^{+\infty} \mu_-(-u) h^-_{\alpha,\beta}(u,z) du.$$

Observing the form of equation (4.3) we see that we have written the characteristic exponent of the Lévy process in a form that resembles the characteristic exponent of a HEJD process. In terms of the parameters $a_i,\ b_i,\ c_i$ and d_i, we have:

$$\begin{aligned} a_i b_i = \omega_i^+ \mu_+(u_i^+) &\quad \text{and} \quad b_i = u_i^+,\ 1 \le i \le N/2 \\ c_i d_i = \omega_i^- \mu_-(-u_i^-) &\quad \text{and} \quad d_i = u_i^-,\ 1 + N/2 \le i \le N. \end{aligned} \tag{4.4}$$

However, there are two sources of error in our proposed approximation:

- The discretization error: In equation (4.2), we replace the integrals by a discrete sum.
- The truncation error: In the third and fourth terms of equation (4.2) respectively, we would like the integrands $\mu_+(u) h^+_{\alpha,\beta}(u,z)$ and $\mu_-(-u) h^-_{\alpha,\beta}(u,z)$ and hence $\Sigma^{(+)2}$ and $\Sigma^{(-)2}$ to be independent of z. However, clearly, neither $\Sigma^{(+)2}$ nor $\Sigma^{(-)2}$ is independent of z and this prevents us from immediately identifying the parameter σ^2 as being equal to $\Sigma^{(+)2} + \Sigma^{(-)2}$.
  Note that for $\nu(x)$ to be a valid Lévy measure, it can easily be shown that $\mu_+(u)$ and $\mu_-(-u)$ must grow less quickly than quadratic as $u \to \infty$ and hence observing the forms of $h^+_{\alpha,\beta}(u,z)$ and $h^-_{\alpha,\beta}(u,z)$, the integrals converge and $\lim_{\overline{\omega}_+ \to \infty} \frac{1}{2}\Sigma^{(+)2} = 0$

and $\lim_{\overline{\omega}_- \to \infty} \frac{1}{2}\Sigma^{(-)2} = 0$. Hence the error in the third and fourth terms could be viewed as a truncation error in the upper limit of the integrals.

Note how, intuitively speaking, these two errors work in opposite directions. Indeed, for a fixed N, if $\overline{\omega}_+$ and $\overline{\omega}_-$ increase, the discretization error gets bigger while the truncation error gets smaller. If $\overline{\omega}_+$ and $\overline{\omega}_-$ decrease, the discretization error gets smaller while the truncation error gets bigger.

We will analyse the discretization error and the truncation error in the next two sections.

4.2 Truncation error

Let us now have a closer look at the term $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + i\alpha z)$. We momentarily view this term as the characteristic exponent of a random variable. We can then calculate the second and third central moments of this random variable by differentiating the term twice or three times.
The second central moment is:

$$\mu_2 \equiv 2\int_{\overline{\omega}_+}^{+\infty} \mu_+(u)\left(\frac{1}{u^3}\right) du + 2\int_{\overline{\omega}_-}^{+\infty} \mu_-(-u)\left(\frac{1}{u^3}\right) du$$

and the third central moment is:

$$\mu_3 \equiv 6\int_{\overline{\omega}_+}^{+\infty} \mu_+(u)\left(\frac{1}{u^4}\right) du - 6\int_{\overline{\omega}_-}^{+\infty} \mu_-(-u)\left(\frac{1}{u^4}\right) du.$$

In practice, the third central moment, μ_3, of $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + i\alpha z)$ should be small in magnitude relative to μ_2. In fact, for a symmetrical Lévy process (ie with $\mu_+(u) = \mu_-(-u)$ and if we were to choose $\overline{\omega}_+ = \overline{\omega}_-$) μ_3 would be identically equal to zero. In any event, it is certainly true that if $\overline{\omega}_+$ and $\overline{\omega}_-$ are large enough, then $|\mu_3| \ll \mu_2$.
Furthermore, if we consider the special case $\alpha = 1$ (i.e. the mean-corrected characteristic exponent) and if we consider the form of $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + iz)$ when $\overline{\omega}_+$ and $\overline{\omega}_-$ are both much, much larger than $|z|$, we see that $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + iz)$ behaves asymptotically like $(z^2 + iz)\left[\int_{\overline{\omega}_+}^{\infty} \mu_+(u)\left(\frac{1}{u^3}\right)\, du + \int_{\overline{\omega}_-}^{\infty} \mu_-(-u)\left(\frac{1}{u^3}\right)\, du\right] = \frac{1}{2}\mu_2(z^2 + iz)$ which we recognize as the mean-corrected characteristic exponent of Brownian motion with variance μ_2.

In order for us to justify the approximation suggested by equation (4.3), we would want the term $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + i\alpha z)$ to be approximately of the form $k(z^2 + i\alpha z)$, where k is independent of z. When $z = 0$, both $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + i\alpha z)$ and $k(z^2 + i\alpha z)$ equal zero and hence there is no approximation. This suggests that, in order to get a handle on the truncation error, we need to compare $\frac{1}{2}\left(\Sigma^{(+)2} + \Sigma^{(-)2}\right)(z^2 + i\alpha z)$ and $k(z^2 + i\alpha z)$ when evaluated at some value of z, z_{large} say, such that $|z|$ is large. Once we have chosen z_{large}

(see section 5), this suggest a measure of the truncation error:

$$\begin{aligned} TE &\equiv \left| \frac{1}{2}\mu_2(z_{large}^2 + i\alpha z_{large}) - \left(\frac{1}{2}\Sigma^{(+)2} \;+\; \frac{1}{2}\Sigma^{(-)2}\right)(z_{large}^2 + i\alpha z_{large}) \right| \\ &= \left| z_{large}^2 + i\alpha z_{large} \right| \left| \int_{\varpi_+}^{+\infty} \mu_+(u)\frac{1}{u^3} du + \int_{\varpi_-}^{+\infty} \mu_-(-u)\frac{1}{u^3}\, du \right. \\ &- \left. \int_{\varpi_+}^{+\infty} \mu_+(u) h^+_{\alpha,\beta}(u, z_{large}) du - \int_{\varpi_-}^{+\infty} \mu_-(-u) h^-_{\alpha,\beta}(u, z_{large})\, du \right| \\ &= \left| z_{large}^2 + i\alpha z_{large} \right| \left| \int_0^{\frac{1}{\varpi_+}} \mu_+\left(\frac{1}{t}\right) t\, dt + \int_0^{\frac{1}{\varpi_-}} \mu_-\left(-\frac{1}{t}\right) t\, dt \right. \\ &- \left. \int_0^{\frac{1}{\varpi_+}} \mu_+\left(\frac{1}{t}\right) h^+_{\alpha,\beta}\left(\frac{1}{t}, z_{large}\right) \frac{1}{t^2}\, dt \;-\; \int_0^{\frac{1}{\varpi_-}} \mu_-\left(-\frac{1}{t}\right) h^-_{\alpha,\beta}\left(\frac{1}{t}, z_{large}\right) \frac{1}{t^2}\, dt \right|, \end{aligned}$$

where we have switched variable to $t = \frac{1}{u}$, in order to help evaluate these integrals (since we will have to compute them numerically and we want to avoid infinite limits).
We will indicate how to choose z_{large} in section 5.

4.3 Discretization error

The discretization error can be estimated quite easily. We estimate our integrals $\int_0^{\varpi_+} \mu_+(u) g^+_{\alpha,\beta}(u,z) du$ and $\int_0^{\varpi_-} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\, du$ using numerical methods such as Gaussian quadrature or Gauss Lobatto quadrature, using some number of points N_{large} where $N_{large} \gg N$.
We have to choose a value of z at which the integrals are computed. From the definitions of $g^+_{\alpha,\beta}(u,z)$ and $g^-_{\alpha,\beta}(u,z)$, we know that when $z = 0$, $g^+_{\alpha,\beta}(u,0)$ and $g^-_{\alpha,\beta}(u,0)$ are both identically equal to zero for all u.
Hence, in order to get a meaningful estimate of the discretization error, we need to evaluate the integrands at some value of z such that $|z|$ is large. We elect to evaluate them at the same z_{large} that we used in estimating the truncation error in the previous section.
Hence, we get an estimate for the discretization error.
For the up jumps, it is:

$$\begin{aligned} DE^+ &\equiv \sum_{i=1}^{N/2} \omega_i^+ \mu_+(u_i^+) g^+_{\alpha,\beta}(u_i^+, z_{large}) - \int_0^{\varpi_+} \mu_+(u) g^+_{\alpha,\beta}(u, z_{large})\, du \\ &\approx \sum_{i=1}^{N/2} \omega_i^+ \mu_+(u_i^+) g^+_{\alpha,\beta}(u_i^+, z_{large}) - \sum_{j=1}^{N_{large}} \omega_j^+ \mu_+(u_j^+) g^+_{\alpha,\beta}(u_j^+, z_{large}). \end{aligned}$$

Similarly, for the down jumps, it is:

$$\begin{aligned} DE^- &\equiv \sum_{i=1+N/2}^{N} \omega_i^- \mu_-(-u_i^-) g^-_{\alpha,\beta}(u_i^-, z_{large}) - \int_0^{\varpi_-} \mu_-(-u) g^-_{\alpha,\beta}(u, z_{large})\, du \\ &\approx \sum_{i=1+N/2}^{N} \omega_i^- \mu_-(-u_i^-) g^-_{\alpha,\beta}(u_i^-, z_{large}) - \sum_{j=1}^{N_{large}} \omega_j^- \mu_-(-u_j^-) g^-_{\alpha,\beta}(u_j^-, z_{large}). \end{aligned}$$

The total discretization error DE is the modulus of the sum of the discretization errors for the up and down jumps. Hence the discretization error is $DE \equiv |DE^+ + DE^-|$.

4.4 Initial estimates

We have examined the forms of the discretization error and the truncation error. Now we need a way to choose the limits $\overline{\omega}_+$ and $\overline{\omega}_-$ of the integrals. Since the errors act in opposite directions, a possible criterion is to find $\overline{\omega}_+$ and $\overline{\omega}_-$ such that the discretization error and the truncation error are equal, using for example a solver or a root finder method.

Using a solver or a root finder method does not usually guarantee that the algorithm finds the global minimum rather than a local minimum, which would contradict the hypothesis that our model satisfies the feature 1 (no non-linear least-squares fitting is required) presented in the introduction.
However, in this case, we conjecture that the two errors are monotonic: when $\overline{\omega}_+$ and $\overline{\omega}_-$ increase (resp. decrease), DE gets larger (resp. smaller) and TE gets smaller (resp. larger). Thus there is only one minimum, and this will ensure that when the solver return a minimum, it is the global minimum and hence we obtain the unique roots of the equation $TE = DE$. Numerical experimentation appears to support our view that the equation $TE = DE$ has unique roots although we do not have a formal proof in full generality.

Once we get $\overline{\omega}_+$ and $\overline{\omega}_-$ such that the errors are equal, we can get ω_i^+ and u_i^+, for $i = 1, ..., N/2$, from a $N/2$-point Gaussian quadrature rule on the interval $(0, \overline{\omega}_+)$ and likewise we can get ω_i^- and u_i^-, for $i = 1+N/2, ..., N$, from a $N/2$-point Gaussian quadrature rule on the interval $(0, \overline{\omega}_-)$ (see Appendix A).

We can then immediately get estimates for a_i, b_i, for $1 \leq i \leq N/2$, c_i, d_i, for $1 + N/2 \leq i \leq N$ and σ^2:

$$\begin{aligned} b_i = u_i^+ \quad &\text{and} \quad d_i = u_i^- \\ a_i = \frac{\omega_i^+ \mu_+(u_i^+)}{b_i} \quad &\text{and} \quad c_i = \frac{\omega_i^- \mu_-(-u_i^-)}{d_i} \\ \sigma^2 \quad &= \quad \mu_2. \end{aligned} \tag{4.5}$$

We now have initial estimates for a_i, b_i, c_i, d_i and σ^2. Our algorithm will work with any Lévy process and, hence, is quite general. In the next section, we show that the algorithm can be simplified for the case of a CGMY process.

4.5 Algorithm for CGMY

In this section, we will show that for the CGMY process, the algorithm can be simplified, since the parameters M and G provide natural scalings.
Recall

$$\mu_+(u) = C\frac{(u-M)^Y}{\Gamma(1+Y)} \mathbf{1}_{\{u>M\}}$$

and

$$\mu_-(-u) = C\frac{(u-G)^Y}{\Gamma(1+Y)} \mathbf{1}_{\{u>G\}}.$$

These natural scalings allow us to simplify the previous algorithm.

Without a great loss of generality, we will take $\alpha = 0$ and $\beta = 1$ (the other cases can be handled by analogy). If we change the variables in equation (4.2), by putting

$$p = u - M \quad \text{and} \quad q = u - G,$$

the characteristic function becomes

$$\begin{aligned}
\phi(z) \;=\; & \int_0^{\overline{\theta}_+} \frac{Cp^Y}{\Gamma(1+Y)} \left(\frac{1}{p+M-iz} - \frac{1}{p+M} - \frac{iz}{(p+M)^2} \right) dp \\
& + \int_0^{\overline{\theta}_-} \frac{Cq^Y}{\Gamma(1+Y)} \left(\frac{1}{q+G+iz} - \frac{1}{q+G} + \frac{iz}{(q+G)^2} \right) dq \\
& - z^2 \int_{\overline{\theta}_+}^{+\infty} \frac{Cp^Y}{\Gamma(1+Y)} \left(\frac{p+M+iz}{(p+M)^2((p+M)^2+z^2)} \right) dp \\
& - z^2 \int_{\overline{\theta}_-}^{+\infty} \frac{Cq^Y}{\Gamma(1+Y)} \left(\frac{q+G-iz}{(q+G)^2((q+G)^2+z^2)} \right) dq
\end{aligned}$$

where $\overline{\theta}_+ = \overline{\omega}_+ - M$ and $\overline{\theta}_- = \overline{\omega}_- - G$.

Given the natural scalings provided by the parameters M and G, it seems reasonable to assume that the upper limits in the integrals with respect to p and to q are the same, that is $\overline{\theta} \equiv \overline{\theta}_+ = \overline{\theta}_-$.
With the change of variable $t = \frac{1}{p}$ and $t = \frac{1}{q}$ in the two last integrals, the characteristic function becomes:

$$\begin{aligned}
\phi(z) \;=\; & \int_0^{\overline{\theta}} \frac{Cp^Y}{\Gamma(1+Y)} \left(\frac{1}{(p+M)-iz} - \frac{1}{p+M} - \frac{iz}{(p+M)^2} \right) dp \\
& + \int_0^{\overline{\theta}} \frac{Cq^Y}{\Gamma(1+Y)} \left(\frac{1}{(q+G)+iz} - \frac{1}{q+G} + \frac{iz}{(q+G)^2} \right) dq \\
& - z^2 \int_0^{\frac{1}{\overline{\theta}}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \left(\frac{Mt+1+izt}{(1+Mt)^2((1+Mt)^2+z^2t^2)} \right) dt \\
& - z^2 \int_0^{\frac{1}{\overline{\theta}}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \left(\frac{Gt+1-izt}{(1+Gt)^2((1+Gt)^2+z^2t^2)} \right) dt.
\end{aligned}$$

As before, we approximate the first two integrals by sums, with weights ω_i and abscissas u_i, for $1 \leq i \leq N/2$, obtained from a $N/2$-point Gaussian quadrature rule on the interval $(0, \overline{\theta})$, with ω_i and u_i, for $1 + N/2 \leq i \leq N$, defined by $\omega_i = \omega_{i-N/2}$ and $u_i = u_{i-N/2}$, and write $\phi(z)$ in the form:

$$\begin{aligned}
\phi(z) \;\simeq\; & \sum_{i=1}^{N/2} \omega_i \frac{Cu_i^Y}{\Gamma(1+Y)} \left(\frac{1}{u_i+M-iz} - \frac{1}{u_i+M} - \frac{iz}{(u_i+M)^2} \right) \\
& + \sum_{i=1+N/2}^{N} \omega_i \frac{Cu_i^Y}{\Gamma(1+Y)} \left(\frac{1}{u_i+G+iz} - \frac{1}{u_i+G} + \frac{iz}{(u_i+G)^2} \right) \\
& - \frac{1}{2}\Sigma^2 z^2.
\end{aligned}$$

The discretization errors are easily determined

$$
\begin{aligned}
DE^{+}_{\overline{\theta}} &= \int_0^{\overline{\theta}} \frac{Cp^Y}{\Gamma(1+Y)} \left(\frac{1}{p+M-iz_{large}} - \frac{1}{p+M} - \frac{iz_{large}}{(p+M)^2} \right) dp \\
&- \sum_{i=1}^{N/2} \omega_i \frac{Cu_i^Y}{\Gamma(1+Y)} \left(\frac{1}{u_i+M-iz_{large}} - \frac{1}{u_i+M} - \frac{iz_{large}}{(u_i+M)^2} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
DE^{-}_{\overline{\theta}} &= \int_0^{\overline{\theta}} \frac{Cq^Y}{\Gamma(1+Y)} \left(\frac{1}{q+G+iz_{large}} - \frac{1}{q+G} + \frac{iz_{large}}{(q+G)^2} \right) dq \\
&- \sum_{i=1+N/2}^{N} \omega_i \frac{Cu_i^Y}{\Gamma(1+Y)} \left(\frac{1}{u_i+G+iz_{large}} - \frac{1}{u_i+G} + \frac{iz_{large}}{(u_i+G)^2} \right)
\end{aligned}
$$

and the total discretization error is given by $DE_{\overline{\theta}} = \left| DE^{+}_{\overline{\theta}} + DE^{-}_{\overline{\theta}} \right|$.

By analogy with section 4.2, the truncation error is given by

$$
\begin{aligned}
TE_{\overline{\theta}} &= |z^2_{large}| \left| \int_0^{\frac{1}{\theta}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \frac{1}{(1+Mt)^3} \, dt + \int_0^{\frac{1}{\theta}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \frac{1}{(1+Gt)^3} \, dt \right. \\
&- \int_0^{\frac{1}{\theta}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \left(\frac{Mt+1+iz_{large}t}{(1+Mt)^2((1+Mt)^2+z^2_{large}t^2)} \right) dt \\
&\left. - \int_0^{\frac{1}{\theta}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \left(\frac{Gt+1-iz_{large}t}{(1+Gt)^2((1+Gt)^2+z^2_{large}t^2)} \right) dt \right| .
\end{aligned}
$$

To make these two errors equal, we no longer need to use a multidimensional root-finder method or a solver, a simple binary search will allow us to compute the unique root $\overline{\theta}$ such that $DE_{\overline{\theta}} = TE_{\overline{\theta}}$. This gives us the mean jump sizes $\frac{1}{b_i}$ and $\frac{1}{d_i}$, initial estimates for the intensities rates a_i and c_i, and for the variance σ^2 of the approximating HEJD process

$$
\begin{aligned}
b_i = M + u_i \quad &\text{and} \quad d_i = G + u_i \\
a_i = \frac{\omega_i}{b_i} \frac{Cu_i^Y}{\Gamma(1+Y)} \quad &\text{and} \quad c_i = \frac{\omega_i}{d_i} \frac{Cu_i^Y}{\Gamma(1+Y)}
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 &= 2 \int_0^{\frac{1}{\theta}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \frac{1}{(1+Mt)^3} \, dt \\
&+ 2 \int_0^{\frac{1}{\theta}} \frac{Ct^{1-Y}}{\Gamma(1+Y)} \frac{1}{(1+Gt)^3} \, dt.
\end{aligned}
$$

We can observe that the mean jump sizes are scaled by M for the up jumps, and by G for the down jumps, which seems a very intuitive result.

4.6 Refining the results

Let us summarize our proposed algorithm up to this point.
We have a robust and simple algorithm which determines essentially (for both up and down

jumps) the jump sizes at which we approximate the small jumps of the Lévy process in question by Brownian motion.

Using standard results from Gaussian quadrature we have estimates for a_i, b_i, c_i, d_i and σ^2 which are essentially analytic. We will see in section 5 that these estimates would allow us to compute prices of vanilla options under a HEJD process which are quite close to the prices of vanilla options under the Lévy process in equation. However, the prices are not as close as we would like. Therefore, we now seek to refine our parameter estimates. We do not refine our estimates of the mean jump sizes $\frac{1}{b_i}$ and $\frac{1}{d_i}$. We regard these as now fixed. We will now denote the initial estimates of a_i, c_i and σ^2 obtained in the last two sections by $a_i^{(0)}$, $c_i^{(0)}$ and $\sigma_i^{(0)2}$.
We now seek to refine our initial estimates of the intensity rates a_i, c_i and the variance σ^2 (to be precise,we allow for the flexibility to keep our initial estimate for $\sigma^{(0)2}$ or to refine it). We will do this by finding those values of a_i, $1 \leq i \leq N/2$, c_i, $1 + N/2 \leq i \leq N$, and σ^2 which allow us to most closely match (in both real and imaginary parts) the characteristic exponent of a HEJD process (multiplied by a carefully chosen weighting function) with the characteristic exponent of the Lévy process in question (multiplied by the same weighting function) at some carefully chosen points z_k, $k = 1, ..., m$, in the complex plane. We denote the weighting function by $\Omega(z)$.

An important point to notice is that the characteristic exponent of the HEJD process is linear in a_i, c_i and σ^2. Indeed, this is the very reason why we choose to fit the characteristic exponents (multiplied by a weighting function) rather than the characteristic functions.

Essentially, we now have to solve a linear system of the form $Ax = b$, where $A \in \mathbb{R}^{2m \times N+1}$, $x \in \mathbb{R}^{N+1}$ and $b \in \mathbb{R}^{2m}$, with $A = [A_{ij}]_{2m \times N+1}$ where, for $1 \leq k \leq m$ and $1 \leq j \leq N+1$,

$$A_{2k-1,j} = \begin{cases} \mathrm{Re}\left[\Omega(z_k)\left(\left(\dfrac{b_j}{b_j - iz_k} - 1\right) - i\alpha z_k\left(\dfrac{b_j}{b_j - 1} - 1\right)\right)\right] & \text{if } 1 \leq j \leq \frac{N}{2} \\ \mathrm{Re}\left[\Omega(z_k)\left(\left(\dfrac{d_j}{d_j + iz_k} - 1\right) - i\alpha z_k\left(\dfrac{d_j}{d_j + 1} - 1\right)\right)\right] & \text{if } \frac{N}{2} + 1 \leq j \leq N \\ \mathrm{Re}\left[-\Omega(z_k)\left(\dfrac{z_k^2 + i\alpha z_k}{2}\right)\right] & \text{if } j = N + 1 \end{cases}$$

$$A_{2k,j} = \begin{cases} \mathrm{Im}\left[\Omega(z_k)\left(\left(\dfrac{b_j}{b_j - iz_k} - 1\right) - i\alpha z_k\left(\dfrac{b_j}{b_j - 1} - 1\right)\right)\right] & \text{if } 1 \leq j \leq \frac{N}{2} \\ \mathrm{Im}\left[\Omega(z_k)\left(\left(\dfrac{d_j}{d_j + iz_k} - 1\right) - i\alpha z_k\left(\dfrac{d_j}{d_j + 1} - 1\right)\right)\right] & \text{if } \frac{N}{2} + 1 \leq j \leq N \\ \mathrm{Im}\left[-\Omega(z_k)\left(\dfrac{z_k^2 + i\alpha z_k}{2}\right)\right] & \text{if } j = N + 1 \end{cases}$$

and

$$x = \begin{bmatrix} a_1 \\ \vdots \\ a_{N/2} \\ c_{1+N/2} \\ \vdots \\ c_N \\ \kappa\sigma^2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \mathrm{Re}\,(\Omega(z_1)(\phi_\alpha(z_1)) + (\kappa-1)\mathrm{Re}\left(\Omega(z_1)\frac{1}{2}\sigma^2(z_1^2 + i\alpha z_1)\right)) \\ \mathrm{Im}\,(\Omega(z_1)(\phi_\alpha(z_1)) + (\kappa-1)\mathrm{Im}\left(\Omega(z_1)\frac{1}{2}\sigma^2(z_1^2 + i\alpha z_1)\right)) \\ \vdots \\ \mathrm{Re}\,(\Omega(z_m)(\phi_\alpha(z_m)) + (\kappa-1)\mathrm{Re}\left(\Omega(z_m)\frac{1}{2}\sigma^2(z_m^2 + i\alpha z_m)\right)) \\ \mathrm{Im}\,(\Omega(z_m)(\phi_\alpha(z_m)) + (\kappa-1)\mathrm{Im}\left(\Omega(z_m)\frac{1}{2}\sigma^2(z_m^2 + i\alpha z_m)\right)) \end{bmatrix}, \tag{4.6}$$

where $\phi_\alpha(z)$ is the characteristic exponent of the Lévy process we are trying to approximate. The roles of α and κ are explained below.
However, if we try to solve this linear system directly, we will have two problems.
Firstly, since we fit both the real and imaginary part of the characteristic exponents, we will have an even number of equations $2m$. If we decide to fit $a_i,\ 1 \leq i \leq N/2,\ c_i,\ 1 + N/2 \leq i \leq N$ and σ^2 the number of parameters will be odd, and the linear system will not be square. In any event, we may wish to have the flexibility to choose m such that $2m > N+1$.
Secondly, solving the linear system directly does not guarantee that $a_i,\ c_i$ and σ^2 are all positive.

To allow us to fit the characteristic exponents at a number of points m such that $2m > N + 1$ if we wish and to try to ensure $a_i,\ c_i$ and σ^2 are all positive, we will use Tikhonov regularization (see Appendix B), with our vector of initial estimates $a_i^{(0)}, c_i^{(0)}$ and for the $\sigma^{(0)2}$ as a starting point.

We allow two possibilities in this fitting:

- We can keep our initial estimate for $\sigma^{(0)2}$ or refine it, by choosing $\kappa = 0$ or $\kappa = 1$ respectively.
- We can decide to fit the characteristic exponents, or the mean-corrected characteristic exponents, by choosing $\alpha = 0$ or $\alpha = 1$ respectively .

In the next section, we will make explicit our choices of the points $z_k,\ k = 1, ..., m$ (where we fit the characteristic exponents), our choice of z_{large} (which we use in calculating the truncation and discretization errors) and our choice of the weighting function $\Omega(z)$.

5 Vanilla options

Lewis (2001) introduced a simple formula that computes the price of a vanilla option for any Lévy process whose characteristic function is known (see also Lipton (2001) and Sepp (2003)). In Sepp (2003), the value of an option whose payoff function is $min\,(S_T, K)$, where S_T is the asset price at maturity T and K is the strike, is shown to be given by

$$f(S_{t_0}, K, T) = \frac{Ke^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{izk} \frac{\Phi_T(-z)}{z^2 - iz}\, dz, \tag{5.1}$$

where $k = \ln\left(\frac{K}{S}\right) - (r-q)T$, Φ_T is the mean-corrected characteristic function of the Lévy process, ν is the imaginary part of z and $\nu \,\epsilon\, (0,1)$.
We can deduce immediately the call and put option prices:

$$C(S,K,T) = Se^{-qT} - f(S,K,T)$$

and

$$P(S,K,T) = Ke^{-rT} - f(S,K,T),$$

where r is the interest rate and q is the dividend yield.
In both Lewis (2001) and Sepp (2003), the integral is evaluated at $\nu = 1/2$, which is symmetrically located between 0 and 1. This gives $z = u + i/2$, where u is real, that we substitute into equation (5.1).
The pricing formula for a call option becomes:

$$C(S,K,T) = Se^{-qT} - \frac{1}{\pi}\sqrt{SK}e^{-\frac{T}{2}(r+q)} \int_0^{+\infty} \mathrm{Re}\left(e^{iuk}\frac{\Phi_T(-u-i/2)}{u^2+1/4}\right) du, \quad . \tag{5.2}$$

Similarly, the put option formula is

$$P(S,K,T) = Ke^{-rT} - \frac{1}{\pi}\sqrt{SK}e^{-\frac{T}{2}(r+q)} \int_0^{+\infty} \mathrm{Re}\left(e^{iuk}\frac{\Phi_T(-u-i/2)}{u^2+1/4}\right) du. \tag{5.3}$$

5.1 Determination of the parameters

So far we still have not determined how to choose some parameters, such as the points z_k where we fit the characteristic exponents, the parameter z_{large} we use to calculate the discretization and the truncation errors or the form of the weighting function $\Omega(z)$.
We will use the form of the integrals in equations (5.1) and (5.2) to motivate our choices. In this dissertation, we would like to compute barrier option prices under the Lévy process in question by approximating the Lévy process by a HEJD process. As a first step to doing this, we need to be able to closely match vanilla prices and hence also the value of the integrals in equations (5.1) and (5.2) computed when $\Phi_T(z)$ is the characteristic function of the Lévy process and when it is the characteristic function of the approximating HEJD process.

5.1.1 Determination of the points z_k

The form of the integrals in equations (5.1) and (5.2) suggests a way of choosing the points z_k, for $k = 1, ..., m$, where we try to match the characteristic exponents. Firstly, we are lead to consider choosing points of the form $z = \pm(u + i/2)$, where u is real. If we momentarily disregard the term e^{iuk} which involves the strike (or equivalently consider a strike K such that $k = 0$), the integral in equation (5.2) has the form

$$\int_0^{+\infty} \text{Re}\left(\frac{\Phi_T(-u - i/2)}{u^2 + 1/4}\right) du = \int_0^{\pi} \text{Re}\left(\Phi_T\left(-\frac{1}{2}\tan\left(-\frac{y}{2}\right) - \frac{i}{2}\right)\right) dy,$$

where we have made the substitution $y = 2\arctan(2u)$.
To price vanilla options we need to evaluate this integral, so a natural way of choosing the points z_k is to choose them so that if we were to have chosen these points z_k as abscissas in the numerical evaluation of this integral, the integral would be as precise as possible. We can again use Gaussian quadrature, to find the abscissas at which we would evaluate the integral.
Discretizing the integral as a discrete sum, we obtain

$$\int_0^{\pi} \text{Re}\left(\Phi_T\left(-\frac{1}{2}\tan\left(-\frac{y}{2}\right) - \frac{i}{2}\right)\right) dy \quad\approx\quad \sum_{k=1}^{m} \omega_k \text{Re}\left(\Phi_T\left(-\frac{1}{2}\tan\left(-\frac{y_k}{2}\right) - \frac{i}{2}\right)\right),$$

where the weights ω_k and the abscissas y_k come from a m-point Gaussian quadrature rule on the interval $(0, \pi)$. This gives us the points z_k, namely $z_k = -\frac{1}{2}\tan\left(\frac{y_k}{2}\right) - \frac{i}{2}$, where we will choose to do the fitting of the characteristic exponents described in section 4.6. We still have to determine the number of points m we will use. We need m to be greater than $N/2$, but we would like m not to be too large. In practice, we found $m \approx 3N/4$ gives very good results.

5.1.2 Determination of z_{large}

In the previous section, we evaluated the discretization error and the truncation error at some point z_{large} such that $|z_{large}|$ is large. To choose the precise value of z_{large}, we can once again draw intuition from the form of the integrals in equations (5.1) and (5.2) (again, we momentarily disregard the term e^{iuk}):

$$\int_0^{+\infty} \text{Re}\left(\frac{\Phi_T(-u - i/2)}{u^2 + 1/4}\right) du.$$

One possibility would be to choose z_{large} of the form $z_{large} = u_{large} + i/2$, where u_{large} is real. However, we actually choose z_{large} of the form $z_{large} = u_{large}$, where u_{large} is real (and positive). We now have to find u_{large}.
Since the integrand converges to 0 as $u \to \infty$, we choose u_{large} such that the integrand is smaller than some specified threshold. Typical values of this threshold are between $1e^{-4}$ and $1e^{-10}$, depending on how fast the integrand converges to 0. Note that, as an aside, since we expect that $u_{\text{large}} \gg 1/2$, we expect little practical difference between the two possible choices.

5.1.3 Determination of the weighting function $\Omega(z)$

We now have to choose a form for the weighting function $\Omega(z)$ that we used in section 4.6 when trying to match the characteristic exponents. As in the last section, noting that we made the choice $z = u + i/2$, where u is real, in going from equation (5.1) to (5.2), we choose $\Omega(z)$ in the form $\Omega(u + i/2)$.
If we observe the form of the integral in equation (5.2), we see that $\mathrm{Re}\left(\frac{\Phi_T(-u-i/2)}{u^2+1/4}\right)$ tends to zero very rapidly as u tends to infinity and it does so because both $\mathrm{Re}\left(\Phi_T(-u - i/2)\right) \rightarrow 0$ and $\frac{1}{u^2+1/4} \rightarrow 0$ as $u \rightarrow \infty$. This means that the contribution to the integral when evaluating the integrand at large u is negligible compared to the contribution to the integral when evaluating the integrand when u is close to zero. This suggests the use of a weighting function $\Omega(z) \equiv \Omega(u + i/2)$ that also tends to zero as $u \rightarrow \infty$.
If we do not do this (for example, if we were to choose $\Omega(z) = 1$, and noting that the characteristic exponent of the HEJD process contains a term proportional to z^2), then when trying to match the characteristic exponents in equation (4.6) of section 4.6 we, intuitively speaking, give too much weight to values of the characteristic exponent when z is large in modulus.
The observation that the characteristic exponent of the HEJD process contains a term proportional to z^2 leads us to propose a weighting function $\Omega(z) = \frac{1}{z^2-iz}$ which is equivalent to $\Omega(u + i/2) = \frac{1}{u^2+1/4}$.

5.2 Vanilla option prices

In this section we will present our results for vanilla option prices obtained from the approximating HEJD process and compare them to those obtained from the Lévy process we want to approximate.
We consider that the parameters of the Lévy process are known. Using the algorithm described in section 4 and choosing z_{large}, $\Omega(z)$ and z_k, $k = 1, ..., m$ as explained above, we obtain the parameters a_i, b_i, c_i, d_i and σ of the approximating HEJD process. All the parameters were computed in Matlab.
When solving the linear system in equation (4.6), we found by numerical experimentation that we obtained the best results when we kept our initial estimate for $\sigma^{(0)2}$ (i.e. $\kappa = 0$) and when we fitted the mean-corrected characteristic exponents (i.e. $\alpha = 1$). Therefore, all the results we report in sections 5 and 6 will use these choices. For the results presented in this section, we set the number of points z_k, m, equal to 10.
We price vanilla options using equations (5.2) and (5.3). For all options we fixed $S_{t_0} = 100$, $r = 0.05$, $q = 0.02$ and the maturity T is 1 year. We consider a range of strikes of the form $K = Fe^y$ where $F = S_{t_0}e^{(r-q)(T-t_0)}$ and $y = -0.4, -0.3, ...,\ 0.3,\ 0.4$.

5.2.1 Vanilla option prices under CGMY

We priced vanilla options for three sets of CGMY parameters. As in Boyarchenko and Levendorskii (2008), we used $C = 1$, $G = 9$, $M = 8$, but we used three different values of Y: (i) $Y = 0.25$, (ii) $Y = 0.5$, (iii) $Y = 1.25$.
The parameters a_i, b_i, c_i, d_i and σ of the corresponding approximating HEJD processes, with $N = 14$ Poisson processes were computed using the simplified algorithm described in

section 4.5. They are displayed in tables 1, 2 and 3 for cases (i), (ii) and (iii) respectively. The different precisions used are: $5e^{-12}$ for the threshold that determines z_{large}, and $\varepsilon = 1e^{-5}$ in the Tikhonov regularization (see Appendix B).
We remark that as the value of Y increases, the value of σ increases which seems intuitive because we know as Y increases, the number of very small jumps increases which is then (by a central Limit Theorem type of argument) better approximated by Brownian motion. Results for call prices and implied volatilities are displayed in figures 1, 2 and 3.
All graphs show that we obtain very good results. We can also see, particularly on the implied volatility graphs, that using the refined parameters obtained by solving the linear system in equation (4.6) using Tikhonov regularization (labelled *HEJD after regularization*) improves the accuracy of the results compared to using the initial parameter estimates (labelled *HEJD initial*).

5.2.2 Vanilla option prices under NIG

The NIG parameters are those presented in Jeannin and Pistorius (2008): $\alpha = 8.858$, $\beta = -5.808$ and $\delta = 0.174$.
The parameters a_i, b_i, c_i, d_i and σ of the corresponding approximating HEJD processes, with $N = 14$ Poisson processes were computed using the algorithm described in section 4. They are displayed in table 11.
The different precisions used are: $1e^{-5}$ for the threshold that determines z_{large}, and $\varepsilon = 1e^{-5}$ in the Tikhonov regularization .
Results for call prices and implied volatilities are displayed in figure 4. Again we obtain excellent results. We observe that in this case the initial estimates were very close to the NIG values.

5.2.3 Vanilla option prices under VG

The VG parameters are those presented in Jeannin and Pistorius (2008): $C = 0.925$, $M = 11.876$ and $G = 4.667$.
The parameters a_i, b_i, c_i, d_i and σ of the corresponding approximating HEJD processes, with $N = 14$ Poisson processes were computed using the algorithm described in section 4. They are displayed in table 13.
The different precisions used are: $1e^{-4}$ for the threshold that determines z_{large}, and $\varepsilon = 1e^{-5}$ in the Tikhonov regularization .
Results for call prices and implied volatilities are displayed in figure 5. The results are again very good. The results obtained from the initial estimates are very precise, and they get even better after using the refined parameter estimates.

5.3 Vanilla option price comparisons

We will know now compare, in greater detail, vanilla option prices obtained from General Classes of Lévy processes with those obtained from the approximation of the Lévy process by a HEJD process. Specifically, we will compare vanilla option prices in the VG and NIG models obtained by four different approaches. The VG and NIG parameters are as in sections 5.2.2 and 5.2.3 and were obtained by a calibration to the market prices of vanilla options (on the Eurostoxx 50 equity index, see Jeannin and Pistorius (2008)). The four different approaches are:

(a) Using equation (5.1) with the characteristic function for either the VG process or the NIG process. Clearly this approach will give us the "true" values against which we can benchmark the accuracy of approaches (b), (c) and (d).

(b) Using equation (5.1) with the characteristic function for the HEJD process where we have fitted fourteen Poisson processes (seven up and seven down) using the methodology described in section 4.

(c) Using equation (5.1) with the characteristic function for the HEJD process where we have fitted fourteen Poisson processes (seven up and seven down) as described in section 4. This gave us estimates of a_i, b_i, c_i, d_i and σ^2. For the case of the VG process, we then exactly matched the third cumulant (ie the third central moment, which is essentially the skew). We did this by computing revised estimates of the intensity rates, which we denote by a_i^r, for each i, $i = 1, ..., N/2$, and c_i^r, for each i, $i = 1 + N/2, ..., N$, via $a_i^r = a_i(2C/M^3)/(\sum_{i=1}^{N/2}(a_i/b_i^3))$ and $c_i^r = c_i(2C/G^3)/(\sum_{i=1+N/2}^{N}(c_i/d_i^3))$. These revised estimates ensure that the third cumulant for both up and down jumps, and hence, also, the overall third cumulant, of the VG process and of the HEJD process are exactly equal. Note also that the revised estimates preserve positivity. We also wanted to exactly match the variance by computing a revised estimate of the diffusion variance, which we denote by $\sigma^{r\,2}$, so that the variance of the VG process and of the HEJD process were exactly the same. However, we could not do this as it would have implied that $\sigma^{r\,2}$ was very slightly negative. Therefore, on the grounds of pragmatism, we set the revised diffusion variance to be exactly zero ie $\sigma^{r\,2} = 0$. For the case of the NIG process, we only revised our estimate for σ^2 and we did so by exactly matching the variance of the NIG process and of the HEJD process. The revised estimate $\sigma^{r\,2}$ was positive.

(d) Using equation (5.1) with the characteristic function for the HEJD process with fourteen Poisson processes where we have used the intensity rates, mean jump sizes and diffusion volatility from Jeannin and Pistorius (2008). This data was supplied by Marc Jeannin and Martijn Pistorius to whom we again express our thanks.

The VG parameters are: $C = 0.925$, $M = 11.876$ and $G = 4.667$. The NIG parameters are: $\alpha = 8.858$, $\beta = -5.808$ and $\delta = 0.174$.

We valued vanilla options with an initial asset price $S_{t_0} = 100$, risk-free rate $r = 0.0$, dividend yield $q = 0.0$ and time to maturity equal to one year. We priced options with 41 different strikes where the strikes were of the form $100 \exp(y)$ where the value of y ranged from -0.8 to 0.8 in intervals of 0.04. Hence, the strikes varied from approximately 44.93 to approximately 222.55. For the options with strikes greater than or equal to 100, we valued call options, else we valued put options. We then converted these prices to implied volatilities (expressed as percentages). We verified (by increasing the number of points used in the numerical integration and by increasing the value at which we truncated the infinite upper limit in equations (5.2) and (5.3)) that all the implied volatilities (when expressed as percentages) reported were accurate to at least five decimal places.

The results are in tables 15 and 16 and in figures 6 and 7.

We can take approach (a) as giving the "true" values, then we can take the differences in implied volatilities between approach (a) and the other three approaches as being a measure of the error of the methodology for approximating either a VG process or a NIG process by a HEJD process consisting of fourteen Poisson processes. For the case of the VG process, the root-mean-square errors were 0.0479, 0.0541 and 0.0635 (expressed as implied volatility percentage points) for approaches (b), (c) and (d) respectively and the maximum absolute errors across all 41 strikes were 0.0941, 0.1750 and 0.2006 for approaches (b), (c) and (d) respectively. For the case of the NIG process, the root-mean-square errors were 0.2044, 0.1989 and 4.2558 (expressed as implied volatility percentage points) for approaches (b), (c) and (d) respectively and the maximum absolute errors across all 41 strikes were 0.6537, 0.6385 and 10.3291 for approaches (b), (c) and (d) respectively.

We now briefly discuss the results.

Firstly, we discuss the results for the case of the VG process.

Clearly, approaches (b), (c) and (d) all work very well. However, approach (b) is, overall, slightly more accurate than the other two approaches. As a rough guide, bid-offer spreads (in implied volatility terms) for Eurostoxx 50 options are probably around 0.2 percentage points to 0.6 percentage points. Hence, all of the errors in implied volatilities from approach (b) are, at worst, less than half the bid-offer spread and are, on average, around one order of magnitude smaller than the bid-offer spread. Approach (c) does not perform quite as well as approach (b). This shows that exactly matching the third cumulant, though superficially appealing, is, in fact, not particularly successful. The idea of moment matching is roughly equivalent to matching the characteristic function around zero but we know option prices also depend upon the characteristic function away from zero. By attempting to more closely match the characteristic function around zero, we may actually make worse the fit to the characteristic function away from zero. Our results seem to support this. Overall, approaches (b) and (c) both perform a little better than approach (d) although approach (d) has the possible benefit that the graph of the errors (figure 7) is a little smoother than for approaches (b) and (c).

Secondly, we discuss the results for the case of the NIG process.

All three approaches fit quite well for low strikes but the fit is visibly much less good for high strikes. This is particularly the case for approach (d) where the quality of the fit is much, much worse than for approaches (b) and (c). Based on the root-mean-square errors and maximum absolute errors across all 41 strikes, approach (c) works a little better than approach (b). However, inspection of figure 6 shows that, in fact, the slightly better performance of approach (c) is explained by the fact that it performs better at very high strikes. Across the range of strikes from, say, 65.0 to, say, 155.0 (which might be more relevant in practice), approach (b) works better than approach (c). Again, we see that while moment matching (in this case, exactly matching the variance) seems appealing, it is not, in fact, particularly successful.

6 Barrier option price comparisons

We will now proceed to examine and compare barrier option prices obtained from the approximation of the Lévy process by a HEJD process which we described in section 4.

Our approach is as follows. We take as given the parameters of the Lévy process in question. Using the results of section 4, we approximate the Lévy process by a HEJD process (by default, with fourteen Poisson processes - seven producing up jumps and seven producing down jumps - although, in sections 6.4 and 6.5 we will consider different numbers of Poisson processes). We then price barrier options using the methodology of Carr and Crosby (2008). The Carr and Crosby (2008) methodology relies on the fact that the Laplace Transform of the barrier option price can be computed essentially in closed form. The Gaver-Stehfest algorithm is then used to invert the Laplace Transform and obtain the barrier option price. We will use the terminology nTerms to denote the number of terms used in the Gaver-Stehfest algorithm. We will refer to this methodology as the HEJDCC methodology.

In order to give us a benchmark against which to compare our results, we will also utilise another approach. Boyarchenko and Levendorskii (2008a), (2008b) describe (the former for single barrier options, the latter for double barrier options) FFT-based algorithms for pricing barrier options under General Classes of Lévy processes. These algorithms are not based on approximating the Lévy process by a HEJD process. It should be said that the Boyarchenko and Levendorskii (2008a), (2008b) methodology (which we will henceforth refer to as the BoyarLeven methodology) uses numerical methods and hence can't be and won't be literally exact. However, it does not approximate the Lévy process in question by a HEJD process at the outset as our approach does and it does appear to give accurate barrier option prices. Hence, we take as a working hypothesis that the option prices obtained from the BoyarLeven methodology are the most accurate and, therefore, we can and will use prices obtained from the BoyarLeven methodology to benchmark the accuracy of our HEJDCC methodology. These prices were provided to us by Mitya Boyarchenko. We again express our thanks to him and to Sergei Levendorskii.

In section 6.2, we will also compare the prices from the HEJDCC methodology against prices reported in Jeannin and Pistorius (2008). They also price barrier options by approximating the Lévy process in question by a HEJD process but their methodology to do this approximation is very different. We will refer to their methodology as the JPHEJD methodology. We again express our thanks to Marc Jeannin and Martijn Pistorius.

The rest of this section is structured as follows:

In section 6.1, we compare single barrier option prices under the CGMY process for different values of the Y parameter using the HEJDCC and BoyarLeven methodologies. In section 6.2, we compare single barrier option prices under the NIG process using the HEJDCC, BoyarLeven and JPHEJD methodologies. In section 6.3, we compare double barrier option prices under the CGMY process for different values of the Y parameter using the HEJDCC and BoyarLeven methodologies. The above comparisons all use fourteen Poisson processes for the HEJD process. In section 6.4, we compare double barrier option prices under the CGMY process for different values of the Y parameter using the BoyarLeven methodology and using the HEJDCC methodology with different numbers of Poisson processes. In section 6.5, we compare double barrier option prices under the

CGMY process using the BoyarLeven methodology and using the HEJDCC methodology with different numbers of Poisson processes using CGMY parameters obtained by fitting the CGMY process to the market prices of vanilla fx options on cable (USD/STG).

6.1 Single barrier option price comparisons under CGMY

In this section, we will price single barrier options under the CGMY process for different values of the Y parameter using the HEJDCC and BoyarLeven methodologies. The data we will use is closely based on that in table 5 of Boyarchenko and Levendorskii (2008a).

We price down-and-out put barrier options with the barrier set at 2100. If the asset price trades at a level less than or equal to the barrier at any time up to and including maturity, the options are knocked out and expire worthless. If the options are not knocked out, then the options have the same payoff at maturity as vanilla put options with strike 3500. The risk-free rate $r = 0.03$, the dividend yield $q = 0.0$ and all options have a maturity of 0.1 years. We price the options with 68 different initial asset prices which are expressed as a percentage of 3500. The percentages are: 60.1, 60.2,(ie in intervals of 0.1),...,62.0, then 63.0, 64.0, (ie in intervals of 1.0),...,110.0. We have smaller intervals when the initial asset price is close to the barrier in order to more closely examine the behaviour of the option prices in this region.

We priced the down-and-out put barrier options for all the different initial asset prices for three different combinations of CGMY parameters. In all three cases, we used $C = 1$, $G = 9$, $M = 8$ in order to match the Boyarchenko and Levendorskii (2008a) data. We varied Y. The three different values of Y were: (1) $Y = 0.25$, (2) $Y = 0.5$ and (3) $Y = 1.25$.

For each combination of CGMY parameters, we fitted a HEJD process with fourteen Poisson processes (seven up and seven down) using the results of section 4. The values of the parameters a_i, b_i, c_i, d_i and σ are displayed in tables 1, 2, 3 .

We then priced the barrier options using the methodology of Carr and Crosby (2008). The Carr and Crosby (2008) paper always works with double barrier options, so we priced the options as if they were double barrier knockout put options and set the upper barrier level to a sufficiently high level that the probability of reaching it would be negligible. We used two different values of nTerms (the number of terms used in the Laplace inversion), namely 12 and 14. We computed the prices of the barrier options with the following four combinations of nTerms and upper barrier levels: nTerms 12 and upper barrier 12000, nTerms 12 and upper barrier 11000, nTerms 14 and upper barrier 11000, nTerms 14 and upper barrier 10000. We used these four different combinations to show that our prices are, to within a reasonable tolerance, independent of the value of nTerms and that we had indeed set the upper barrier level sufficiently high (and the results below support this). As a comparison, we also display the prices of the barrier options using the BoyarLeven methodology.

The results are displayed in tables 17, 18, 19 and in graphical form in figures 8, 9, 10.

Note that the prices for the BoyarLeven methodology in table 18 do not exactly match the prices in table 5 of Boyarchenko and Levendorskii (2008a) (for the case $Y = 0.5$). Mitya Boyarchenko, who kindly provided us with this data, said this is because the computer program was very slightly changed between when table 5 of Boyarchenko and Levendorskii (2008a) was prepared and when he supplied us with this data. The discrepancies are in any event very small.

Overall, the agreement between the prices using the HEJDCC methodology and the

BoyarLeven methodology is very, very good - specially when the initial asset price is not too close to the barrier. In fact, when the initial asset price is greater than, say, about 63.0 per cent of 3500 one has to view figures 8.(a), 9.(a) and 10.(a) extremely closely to see any differences at all between the prices obtained from the BoyarLeven methodology and those obtained from the HEJD methodology (for all four combinations of nTerms and upper barrier levels). However, the agreement does deteriorate somewhat when the initial asset price is extremely close to the barrier.

We remark that the agreement between the prices is just as good for the infinite variation case ($Y = 1.25$) as for the finite variation cases ($Y = 0.25$ and $Y = 0.5$).

6.2 Single barrier option price comparisons under NIG

In this section, we will price single barrier options under the NIG process using the HEJDCC, JPHEJD and BoyarLeven methodologies. The data we will use is the same as in table 1 of Boyarchenko and Levendorskii (2008a) and in table 2 of Jeannin and Pistorius (2008).

We price down-and-out put barrier options with the barrier set at 2100. As in the previous section, the options have a strike of 3500, the risk-free rate $r = 0.03$ and the dividend yield $q = 0.0$. In this section, all options have a maturity of one year. We price the options with 32 different initial asset prices which are expressed as a percentage of 3500. The percentages are: 64.0, 66.0,...,126.0. Hence, the initial asset prices varied from 2240 to 4410.

The NIG parameters are $\alpha = 8.858$, $\beta = -5.808$ and $\delta = 0.174$. We fitted a HEJD process with fourteen Poisson processes (seven up and seven down - this is the same number as Jeannin and Pistorius (2008) used) using the results of section 4. The values of the parameters a_i, b_i, c_i, d_i and σ that we obtained are displayed in table 11.

We then priced the barrier options using the methodology of Carr and Crosby (2008). The Carr and Crosby (2008) paper always works with double barrier options, so we priced the options as if they were double barrier knockout put options and set the upper barrier level to 21000 as we judged this to be a sufficiently high level that the probability of reaching it would be negligible. We used two different values of nTerms, namely 12 and 14. As a comparison, we also display the prices of the barrier options using the BoyarLeven methodology and those from Jeannin and Pistorius (2008)

The results are displayed in table 20 and in graphical form in figure 11.

We make some comments about the results:

Overall, the agreement between the prices using the three different methodologies is good. However, it is clear that the prices obtained from the HEJDCC methodology are closer than those of the JPHEJD methodology to the prices obtained from the BoyarLeven methodology. In fact, over the whole range of initial asset prices, the lines in figure 11 depicting the prices using the BoyarLeven methodology and those using the HEJDCC methodology (for either value of nTerms) essentially lie on top of each other.

We take as our working hypothesis that the BoyarLeven methodology is the most accurate and we compute the root-mean-square proportional error (ie the errors relative to the BoyarLeven prices) for the HEJDCC methodology (with nTerms set to 12) and for the JPHEJD methodology.

The root mean-square errors were (for the HEJDCC methodology) 0.00312 and (for the JPHEJD methodology) 0.04401. Hence, the root-mean-square errors in the HEJDCC methodology are about one-fourteenth the root-mean-square errors in the JPHEJD methodology.

The maximum (in magnitude) proportional error (again, relative to the BoyarLeven prices), across the 32 different prices, is approximately seventeen times larger in the JPHEHD methodology than in the HEJDCC methodology.

We believe that the reason for the better performance of the HEJDCC methodology is that the procedure for fitting a HEJD process to the NIG process is much better.

6.3 Double barrier option price comparisons under CGMY

In this section, we will price double barrier options under the CGMY process for different values of the Y parameter using the HEJDCC and BoyarLeven methodologies. The data we will use is closely based on that in table 1 of Boyarchenko and Levendorskii (2008b).

We price two types of double barrier options, namely double barrier knockout put (henceforth DBKP) options and double-no-touch (henceforth DNT) options. For both types, the lower barrier is 2800 and the upper barrier is 4200. If the asset price trades at a level equal to or outside the barriers at any time up to and including maturity, the options are knocked out and expire worthless. If the options are not knocked out, then the DBKP options have the same payoff at maturity as vanilla put options with strike 3500 and the DNT options pay one unit of account at maturity. The risk-free rate $r = 0.03$, the dividend yield $q = 0.0$ and all options have a maturity of 0.1 years. We price both types of options with 75 different initial asset prices which are expressed as a percentage of 3500. The percentages are: 80.1, 80.2,(ie in intervals of 0.1),...,82.0, then 83.0, 84.0, (ie in intervals of 1.0),...,118.0, then 118.1, 118.2, (ie in intervals of 0.1 again),..., 119.9. We have smaller intervals when the initial asset price is close to the barriers in order to more closely examine the behaviour of the option prices in these regions. The change in intervals at 82.0 and 118.0 is the reason why the graphs (see figures 12, 13 and 14) appear to exhibit slight kinks at 82.0 and 118.0 - had we used constant intervals throughout the range 80.1 to 119.9, these kinks would not be present.

We priced both types of barrier option for all the different initial asset prices for three different combinations of CGMY parameters. In all three cases, we used $C = 1$, $G = 9$, $M = 8$. We varied Y. The three different values of Y were: (1) $Y = 0.25$, (2) $Y = 0.5$ and (3) $Y = 1.25$.

For each combination of CGMY parameters, we fitted a HEJD process with fourteen Poisson processes (seven up and seven down) using the results of section 4. The values of the parameters a_i, b_i, c_i, d_i and σ are displayed in tables 1, 2 and 3 .

We then priced the barrier options using the methodology of Carr and Crosby (2008). We priced these options with two different values of nTerms (the number of terms used in the Laplace inversion), namely 12 and 14. As a comparison, we also display the prices of the barrier options using the BoyarLeven methodology.

The results are displayed in tables 21, 22 and 23 and in graphical form in figures 12, 13 and 14.

We make some comments about the results:

Overall, the agreement between the prices using the HEJDCC methodology and the BoyarLeven methodology is very, very good - specially when the initial asset price is not too close to either barrier. However, the agreement does deteriorate when the initial asset price is very close to either barrier.

We compute the root-mean-square proportional errors (by which we mean proportional to the Boyarchenko and Levendorskii (2008b) prices) between the prices obtained by the two different methodologies, (for nTerms set equal to 14) over all 75 different initial asset prices. The values obtained were:

(1) $Y = 0.25$, for DBKP options 0.0836, for DNT options 0.1130.

(2) $Y = 0.5$, for DBKP options 0.1090, for DNT options 0.1158.

(3) $Y = 1.25$, for DBKP options 0.0605, for DNT options 0.0561.

We see that the infinite variation case ($Y = 1.25$) performs very well.

6.4 Double barrier option price comparisons under CGMY with different numbers of Poisson processes

In this section, we price double barrier knockout put (DBKP) options and double-no-touch (DNT) options almost exactly as in the previous section. The only differences, compared to the previous section, are that we only consider a CGMY process when the parameter $Y = 0.5$ but now we consider four different numbers of Poisson processes. We approximate the CGMY process by a HEJD process with: (a) 4 Poisson processes, (b) 8 Poisson processes, (c) 14 Poisson processes (as in the previous section) and (d) 20 Poisson processes. In each case, half the Poisson processes produce up jumps and half the Poisson processes produce down jumps.

We fitted HEJD processes using the results of section 4. The values of the parameters a_i, b_i, c_i, d_i and σ are displayed in tables 4, 5, 2 and 6.

We then priced the barrier options using the methodology of Carr and Crosby (2008) and with the value of nTerms (the number of terms used in the Laplace inversion) set to 12. As a comparison, we also display the prices of the barrier options using the BoyarLeven methodology.

The results are displayed in table 24 and in graphical form in figure 15.

We compute the sum of squares of the proportional errors (by which we mean proportional to the BoyarLeven prices) between the prices obtained by the BoyarLeven methodology and those obtained by the HEJD methodology for the four different numbers of Poisson processes. We do this over all all 75 different initial asset prices. We then divide this by 75 and take the square root. This gives a root-mean-square proportional error. The values obtained were:

(a) 4 Poisson processes, for DBKP options 0.3199, for DNT options 0.2346.

(b) 8 Poisson processes, for DBKP options 0.1751, for DNT options 0.1617.

(c) 14 Poisson processes, for DBKP options 0.1090, for DNT options 0.1158.

(d) 20 Poisson processes, for DBKP options 0.0869, for DNT options 0.0904.

We then repeated the last set of calculations but this time we only considered the 37 different initial asset prices from 82.0 to 118.0 per cent of 3500. This gives a root-mean-square proportional error over those option prices where the initial asset price is not too close to either barrier.

The values obtained were:

(a) 4 Poisson processes, for DBKP options 0.1502, for DNT options 0.0609.

(b) 8 Poisson processes, for DBKP options 0.0963, for DNT options 0.0494.

(c) 14 Poisson processes, for DBKP options 0.0060, for DNT options 0.0059.

(d) 20 Poisson processes, for DBKP options 0.0041, for DNT options 0.0020.

It is clear that as the number of Poisson processes is increased, the prices of the options obtained by the HEJDCC methodology do converge towards those obtained from the BoyarLeven methodology. The rate of convergence appears to be much faster when we consider options when the initial asset price is not too close to either barrier. When the initial asset price is close to either barrier, we have already seen that the agreement between the prices obtained by the HEJDCC and the BoyarLeven methodologies is not as good. It appears that the rate of convergence as the number of Poisson processes is increased is also not as good.

6.5 DNT price comparisons under CGMY with different numbers of Poisson processes

In this section, we price DNT options using market data for vanilla options on cable (USD/STG) as of 6th July 2007. The market data that we used can be found in Ambrose et al. (2008). Firstly, we calibrated a CGMY process to the market prices of vanilla options with maturities of 6 months, 9 months, one year and two years. We fixed the value of the Y parameter to be $Y = 1.25$. We then searched for those values of C, G and M which minimised the sum of squares of proportional differences between the market and model prices. The following values were obtained:

$C = 0.009541485$, $G = 9.992139358$, $M = 10.757164203$.

We then priced twelve DNT options with the barrier levels and maturities as indicated in table 25. The initial spot fx rate was 2.0060. The interest-rates that we used can be found in Ambrose et al. (2008). The DNT options were priced using the HEJDCC methodology with the number of Poisson processes set to 4, 8, 14 and 20 and also using the BoyarLeven methodology. The results are displayed in tables 7, 8, 9, 10 and 25.

Again, we see that as the number of Poisson processes is increased, the prices of the options obtained by the HEJDCC methodology do converge towards those obtained from the BoyarLeven methodology. The agreement between the prices obtained by the two different methodologies is very good for shorter maturity options but does deteriorate for longer maturity options.

In table 25, we have also displayed the market prices of these DNT options. It is noteworthy to mention that the model prices are very significantly higher (sometimes by a factor of two) than the market prices. This reinforces comments made in Carr and Crosby (2008) that many models cannot successfully be calibrated to the market prices of both barrier and vanilla fx options.

6.6 Summary of barrier option price comparisons

We have shown that, by approximating the Lévy process in question by a HEJD process, we can very accurately price barrier options as long as the initial asset price is not too close to the barrier (or barriers). We have shown that this is true whether the Lévy process has finite or infinite variation. We have illustrated that our methodology for approximating the Lévy process by a HEJD process yields more accurate barrier option prices than the methodology of Jeannin and Pistorius (2008).

However, pricing barrier options by approximating the Lévy process by a HEJD process works less well when the initial asset price is very close to the barrier (or barriers). Boyarchenko and Levendorskii (2002) and Boyarchenko and Levendorskii (2008a) explain the reason for this. Under a Lévy process with no Gaussian component, for barrier options with intrinsic value at the barrier, the first and second partial derivatives of the option price with respect to the initial asset price tend to plus or minus infinity (assuming certain technical regularity conditions apply) as the initial asset price tends to the barrier level - even for options whose maturity is bounded away from zero. By contrast, when a Gaussian component is present (such as for a HEJD process), then for options whose maturity is bounded away from zero, the first and second partial derivatives with respect to the initial asset price are finite. This is a fundamental difference. Since the option price must be exactly the same under both processes if the asset price is exactly equal to the barrier, but the gradient is infinite under one process and finite under the other, it must imply that, if the initial asset price is close to the barrier level, barrier option prices under a Lévy process with no Gaussian component and prices under a process with a Gaussian component must differ. We have seen this in our examples.

We make one further comment about this in the spirit of suggesting ideas for future research. The partial derivative of the option price with respect to the initial asset price is usually termed the delta. However, when asset prices are modelled using General Classes of Lévy processes or jump-diffusion processes, the market is incomplete. Therefore, perfect hedging is not possible and the interpretation of delta as a hedge ratio is no longer valid. Cont and Tankov (2004) (see section 10.4) discuss the idea of mean-variance hedging. They look for a self-financing trading strategy over the lifetime of the option which minimises the terminal hedging error defined as the expected squared error. Under regularity conditions, they derive an expression for the optimal hedge ratio (though we stress it is not a perfect hedge). We would conjecture that the optimal hedge ratio (as opposed to the partial derivative of the option price with respect to the initial asset price) is finite under Lévy processes with no Gaussian component and also finite when a Gaussian component is present (such as for a HEJD process). It would be an interesting topic for future research to see by how much they differ when the Lévy process in question has been approximated by a HEJD process.

7 Conclusions

It is known that General Classes of Lévy processes provide a model that can capture the effect of smiles and skews in implied volatilities, which the standard model of Black and Scholes (1973) and Merton (1973) cannot.

Whilst pricing vanilla options remains straightforward (see Lewis (2001), Sepp (2003) and Lipton (2001)), it is much more difficult to price exotic options, since few, if any, analytical results exist. Hence, techniques for approximating Lévy processes by jump-diffusion processes, which are considerably more tractable and thus facilitate the pricing of exotic options, have been developed in the literature. Asmussen et al. (2007) and Jeannin and Pistorius (2008) use a hyperexponential jump-diffusion (HEJD) process as an approximating process to price barrier options.

In this dissertation, our aim was to focus on the methodology of approximating Lévy processes by hyperexponential jump-diffusion processes. We thus developed an algorithm that determines the parameters of the approximating process in a more systematic way than the procedures in the extant literature and that satisfies the six desirable features presented in the introduction. Indeed, our methodology is intuitive and easy to implement and does not require non-linear least-squares fitting in the determination of the parameters of the HEJD process. Our methodology calculates intensity rates and the mean jump sizes of the HEJD process, thus there is no need to guess the mean jump sizes. Our methodology computes the magnitude of the very small jumps of the Lévy process in question, below which the very small jumps are approximated by Brownian motion.

Moreover, comparing our results to those obtained from existing procedures for approximating Lévy processes by a HEJD process, we demonstrate that our methodology computes more accurate option prices, both for vanilla and barrier options.

Appendix A : Gaussian Quadrature

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The Gaussian quadrature rule gives an approximation of the integral of a weighted function as a weighted sum of function values at optimal abscissas x_i, taken within the domain of integration . A n-point Gaussian quadrature rule integrates exactly all polynomials of order up to $2n-1$. The approximation takes the form

$$\int_\alpha^\beta W(x)f(x)dx \approx \sum_{i=1}^n \omega_i f(x_i).$$

The weights ω_i and nodes x_i depend on the choice of the weight function $W(x)$: the fundamental theorem of Gaussian quadrature states that that the optimal abscissas of the n-point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function $W(x)$.

Interval	Weight function	Orthogonal Polynomial
$(-1,1)$	1	Legendre polynomials
$(-1,1)$	$\frac{1}{\sqrt{1-x^2}}$	Chebyshev polynomials
$(0,+\infty)$	e^{-x}	Laguerre polynomials
$(-\infty,+\infty)$	e^{-x^2}	Hermitte polynomials

In our case, we will use the Gauss-Legendre Quadrature, with $W(x) = 1$ and the domain of integration $[-1,1]$. The abscissas x_i are the roots of the Legendre polynomial of degree n, $P_n(x)$, normalized so that $P_n(1) = 1$. The weights ω_i are given by

$$\omega_i = \frac{2}{(1-x_i^2)(P_n'(x_i))^2}.$$

For an arbitrary interval $[a,b]$ and $W(x) = 1$, the approximation becomes

$$\int_a^b f(y)dy \simeq \sum_{i=1}^n \omega_i' f(y_i)$$

where $\omega_i' = \frac{b-a}{2}\omega_i$ and $y_i = \left(\frac{b-a}{2}\right)x_i + \left(\frac{a+b}{2}\right)$.

Appendix B: Tikhonov Regularization

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Tikhonov regularization, also known in statistics as ridge regression, is used in the case of ill-posed problems. Solving a linear system

$$Ax = b$$

where the matrix A is singular or ill-conditioned is transformed into a minimization, by choice of x, of the term

$$\|Ax - b\|^2 + \|\Gamma x\|^2$$

where Γ is the Tikhonov matrix (e.g. $\Gamma = \alpha I$, where $\alpha \in \mathbb{R}$ and I is the identity matrix), which will give an explicit solution $\overline{x}$. In the specific case where we already have an initial estimate x_0 of potential solutions, we can use generalized Tikhonov regularization, where we seek x that minimizes

$$\|Ax - b\|^2 + \varepsilon^2 \|x - x_0\|^2 , \tag{B.1}$$

where $\varepsilon \in \mathbb{R}^+$. The solution $\overline{x}$ that minimizes $(B.1)$ is given by

$$\overline{x} = x_0 + (A^T A + \varepsilon^2 I)^{-1} A^T (b - Ax_0).$$

Note that when $\varepsilon \to 0$, $\overline{x}$ is the same as we would get from solving $Ax = b$ by singular value decomposition. When $\varepsilon \to \infty$, $\overline{x} \to x_0$. Hence, in our application in section 4.6, we need to find ε that will allow us to reach a compromise between the positivity of the coefficients a_i, c_i and σ^2 and the precision of our solution.

Appendix C : Tables of parameters values

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- Results for CGMY model

a_i	1.376192	0.063706	3.764656	2.380610	1.550909	1.014480	0.607951
b_i	11.813917	27.369977	52.526709	82.941254	113.355798	138.512530	154.068591
c_i	1.283897	1.370712	3.043468	2.054498	1.385181	0.905440	0.514892
d_i	13.290656	30.791224	59.092548	93.308910	127.525273	155.826597	173.327164
σ	0.0116739						

Table 1: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 1,\ G = 9,\ M = 8,\ Y = 0.25$ and $N = 14$.

a_i	1.277247	3.090880	4.490299	3.742768	2.878943	1.931241	0.921386
b_i	11.283771	23.888283	44.523171	69.470795	94.418419	115.053306	127.813212
c_i	1.308219	3.690592	4.420698	3.798812	2.959917	1.986005	0.927167
d_i	12.519424	26.874319	50.088567	78.154644	106.220721	129.434969	143.789864
σ	0.0292297						

Table 2: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 1,\ G = 9,\ M = 8,\ Y = 0.5$ and $N = 14$.

a_i	4.060298	56.167802	79.723611	99.794582	100.529093	78.433287	37.796564
b_i	12.713731	31.939922	63.031864	100.622089	138.212315	169.304257	188.530447
c_i	5.272607	58.584714	90.345791	114.801028	116.034606	90.579939	43.554179
d_i	14.302948	35.932412	70.910846	113.199851	155.488855	190.467289	212.096753
σ	0.2061631						

Table 3: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 1,\ G = 9,\ M = 8,\ Y = 1.25$ and $N = 14$.

a_i	14.447122	0.382358
b_i	25.029509	71.554993
c_i	9.135528	1.987425
d_i	28.158198	80.499367
σ	0.0378816	

Table 4: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 1,\ G = 9,\ M = 8,\ Y = 0.5$, with $N = 4$.

a_i	3.882277	0.135394	2.333864	0.951439
b_i	14.634859	39.535478	72.023833	96.924452
c_i	4.051768	2.204519	3.046327	1.307420
d_i	16.464217	44.477413	81.026812	109.040008
σ	0.0342513			

Table 5: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 1,\ G = 9,\ M = 8,\ Y = 0.5$, with $N = 8$.

a_i	0.733036	2.601613	2.918597	3.007904	2.850838
b_i	10.068499	18.696785	33.414054	52.916250	75.470988
c_i	0.776421	2.766407	3.118385	3.206633	3.032924
d_i	11.327061	21.033883	37.590810	59.530781	84.904862
a_i	2.529817	2.099597	1.593645	1.035947	0.447210
b_i	99.074315	121.629054	141.131250	155.848519	164.476805
c_i	2.688760	2.230612	1.693019	1.100996	0.476315
d_i	111.458605	136.832686	158.772656	175.329583	185.036405
σ	0.0246939				

Table 6: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 1,\ G = 9,\ M = 8,\ Y = 0.5$, with $N = 20$.

a_i	1.1661	1.6523
b_i	39.1204	116.6102
c_i	0.9889	1.5036
d_i	36.3383	108.3172
σ	0.0228286	

Table 7: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 0.00954,\ G = 9.99214,\ M = 10.75716,\ Y = 1.25$ and $N = 4$.

a_i	0.286149	1.530737	1.726521	1.018453
b_i	24.156613	74.444730	140.056679	190.344796
c_i	0.255940	1.451108	1.592568	0.938850
d_i	22.438650	69.150392	130.096170	176.807911
σ	0.0203079			

Table 8: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 0.00954,\ G = 9.99214,\ M = 10.75716,\ Y = 1.25$ and $N = 8$.

a_i	0.106440	1.125995	1.649283	2.063164	2.072079	1.612734	0.774614
b_i	19.401359	54.658975	111.676395	180.610581	249.544767	306.562186	341.819803
c_i	0.093782	1.064477	1.520398	1.888611	1.893507	1.473348	0.708555
d_i	18.021579	50.771754	103.734226	167.765971	231.797715	284.760187	317.510362
σ	0.0168160						

Table 9: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 0.00954,\ G = 9.99214,\ M = 10.75716,\ Y = 1.25$ and $N = 14$.

a_i	0.067971	0.793893	1.357637	1.946936	2.399857
b_i	17.557095	45.921507	94.302689	158.413721	232.559604
c_i	0.062121	0.714533	1.246223	1.780097	2.190977
d_i	16.308475	42.655675	87.596098	147.147700	216.020499
a_i	2.611452	2.529900	2.149854	1.509484	0.685071
b_i	310.152592	384.298476	448.409508	496.790690	525.155102
c_i	2.382926	2.308064	1.961239	1.377145	0.625293
d_i	288.095251	356.968050	416.519652	461.460075	487.807275
σ	0.0144339				

Table 10: Parameters $a_i,\ b_i,\ c_i,\ d_i$ and σ for $C = 0.00954,\ G = 9.99214,\ M = 10.75716,\ Y = 1.25$ and $N = 20$.

- Results for NIG model

a_i	0.000682	0.443363	0.640733	0.682041	0.619466	0.465224	0.243177
b_i	16.124670	22.074237	31.695674	43.328014	54.960353	64.581791	70.531358
c_i	0.228054	0.586196	1.028877	0.982829	0.827779	0.596787	0.306100
d_i	4.586623	10.854142	20.989762	33.243748	45.497735	55.633355	61.900874
σ	0.0392083						

Table 11: Parameters a_i, b_i, c_i, d_i and σ for $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.

a_i	0.000682	0.443363	0.640733	0.682041	0.619466	0.465224	0.243177
b_i	16.124670	22.074237	31.695674	43.328014	54.960353	64.581791	70.531358
c_i	0.228054	0.586196	1.028877	0.982829	0.827779	0.596787	0.306100
d_i	4.586623	10.854142	20.989762	33.243748	45.497735	55.633355	61.900874
σ	0.0406245						

Table 12: Parameters a_i, b_i, c_i, d_i and σ for $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$, where the variance is fitted exactly, and $N = 14$.

- Results for VG model

a_i	0.435258	0.329414	0.177605	0.173601	0.132657	0.074149	0.009247
b_i	14.126684	23.306687	38.152278	56.100629	74.048980	88.894570	98.074574
c_i	0.549198	0.239322	1.106979	0.464675	0.220016	0.101674	0.017561
d_i	5.916007	11.010407	19.248899	29.209254	39.169608	47.408100	52.502501
σ	0.0141106						

Table 13: Parameters a_i, b_i, c_i, d_i and σ for $C = 0.925$, $G = 4.667$, $M = 11.876$ and $N = 14$.

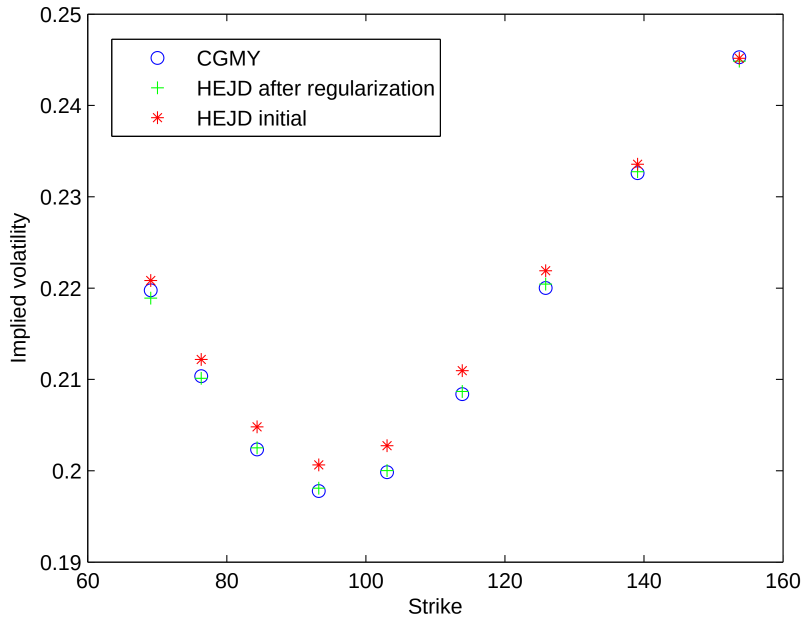
a_i	0.433014	0.327715	0.176690	0.172706	0.131973	0.073766	0.009199
b_i	14.126684	23.306687	38.152278	56.100629	74.048980	88.894570	98.074574
c_i	0.553381	0.241145	1.115411	0.468214	0.221692	0.102448	0.017694
d_i	5.916007	11.010407	19.248899	29.209254	39.169608	47.408100	52.502501
σ	0						

Table 14: Parameters a_i, b_i, c_i, d_i and σ for $C = 0.925$, $G = 4.667$, $M = 11.876$, where the third cumulant is fitted exactly and $N = 14$.

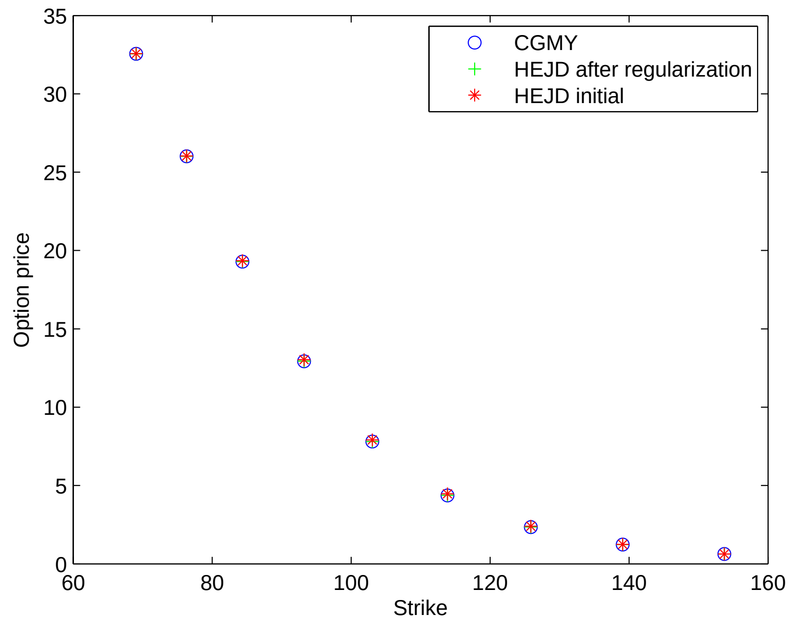
Appendix D : Vanilla option price results

addcontentslinetocchapterAppendix D: Vanilla option price results

- Results for $S_{t_0} = 100$, $r = 0.05$, $q = 0.02$ and $T = 1$.

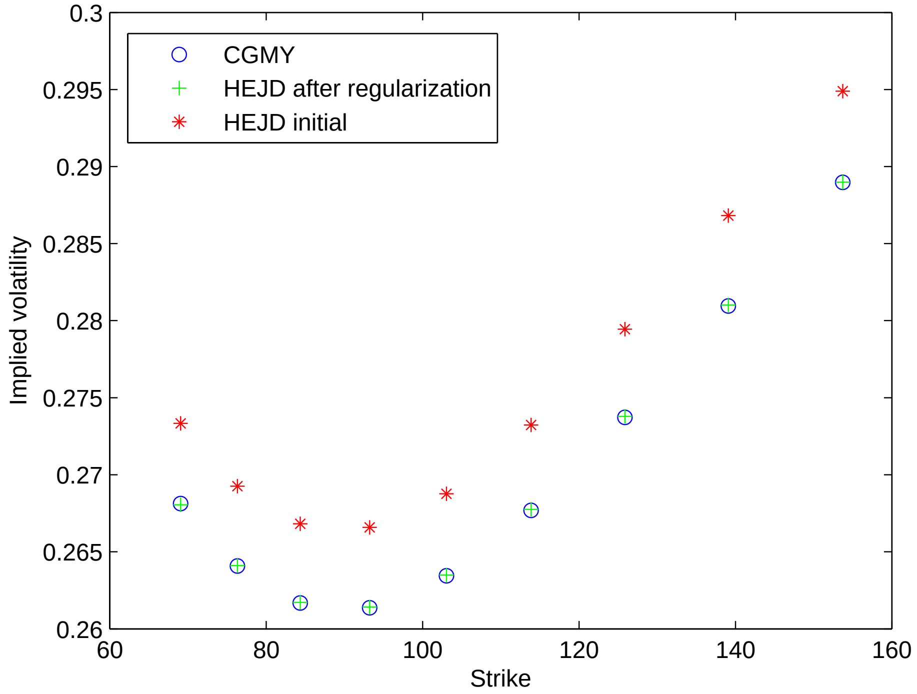


(a) Implied Volatility

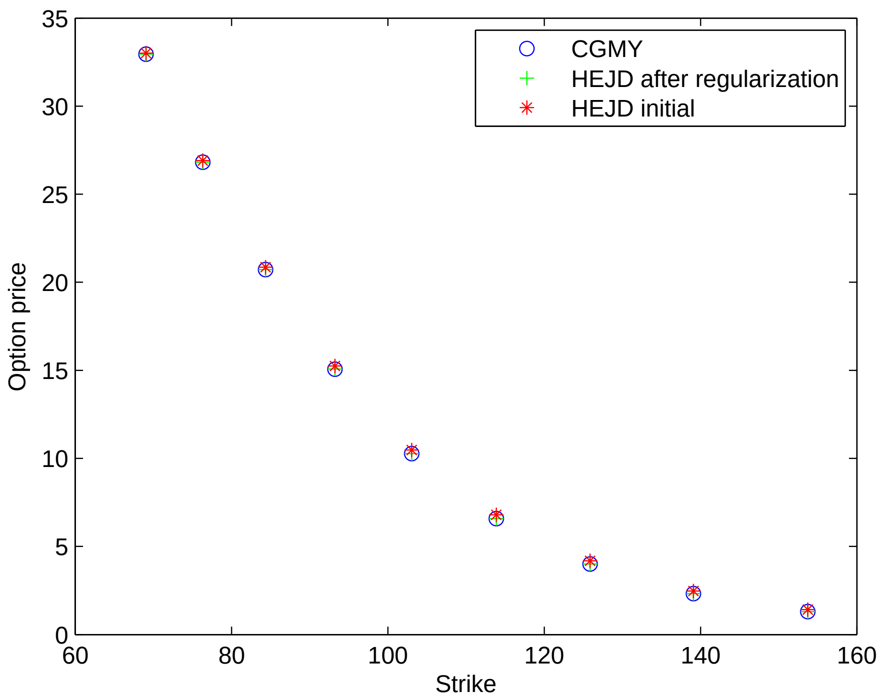


(b) Call prices

Figure 1: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.25$ and $N = 14$.

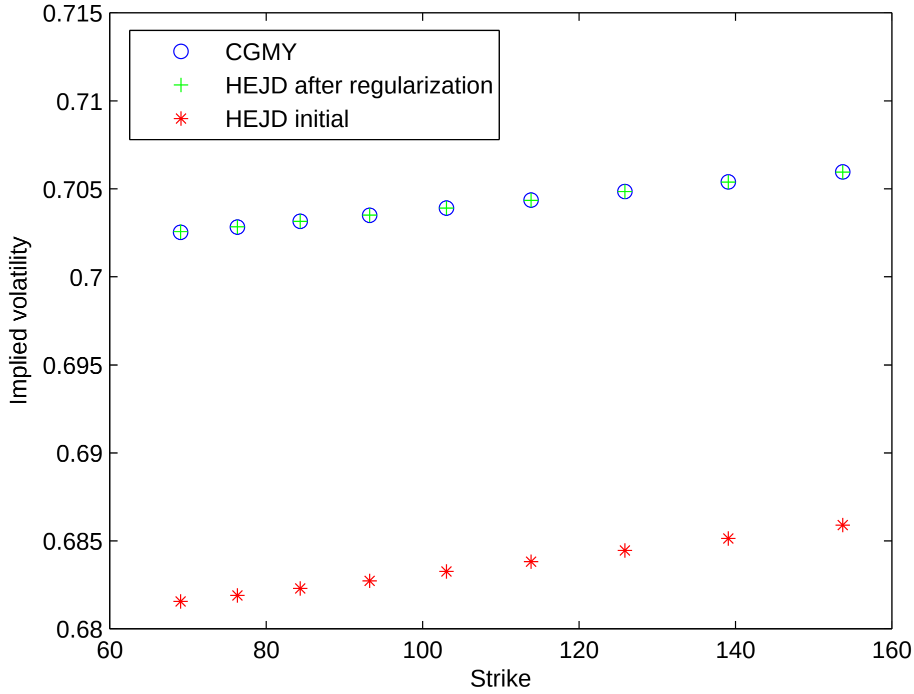


(a) Implied Volatility

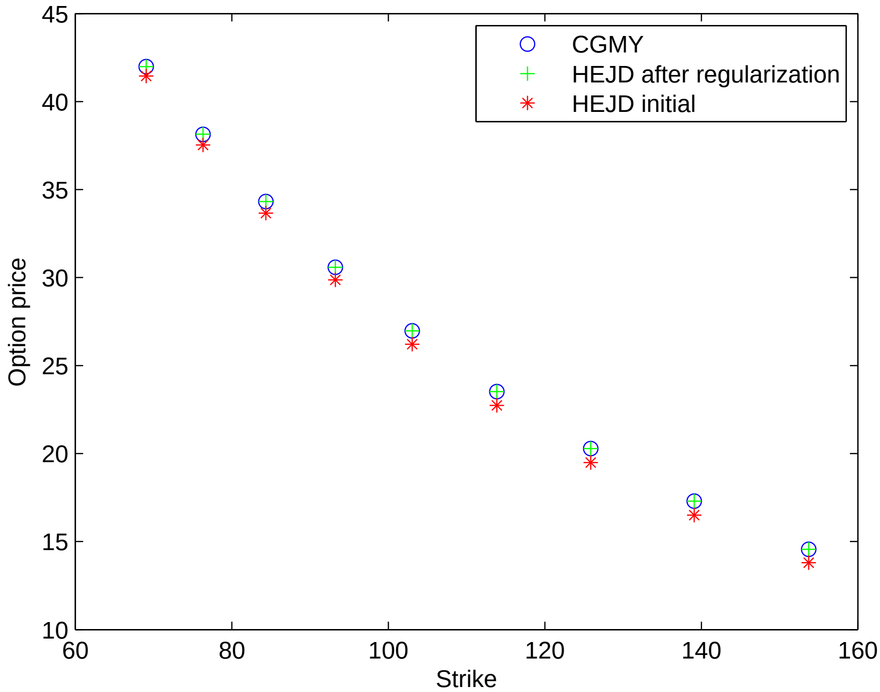


(b) Call prices

Figure 2: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and $N = 14$.

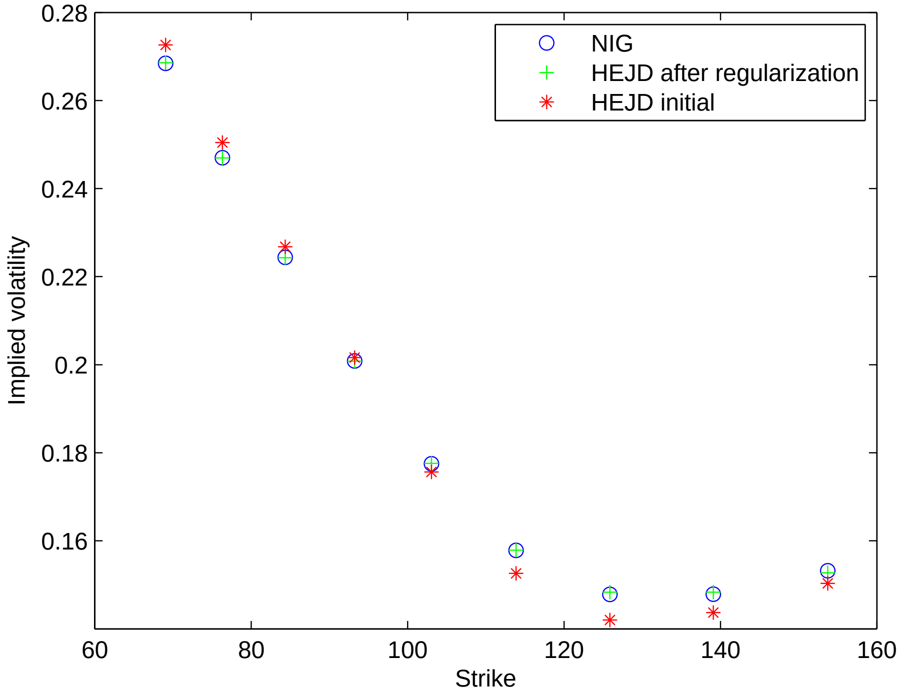


(a) Implied Volatility

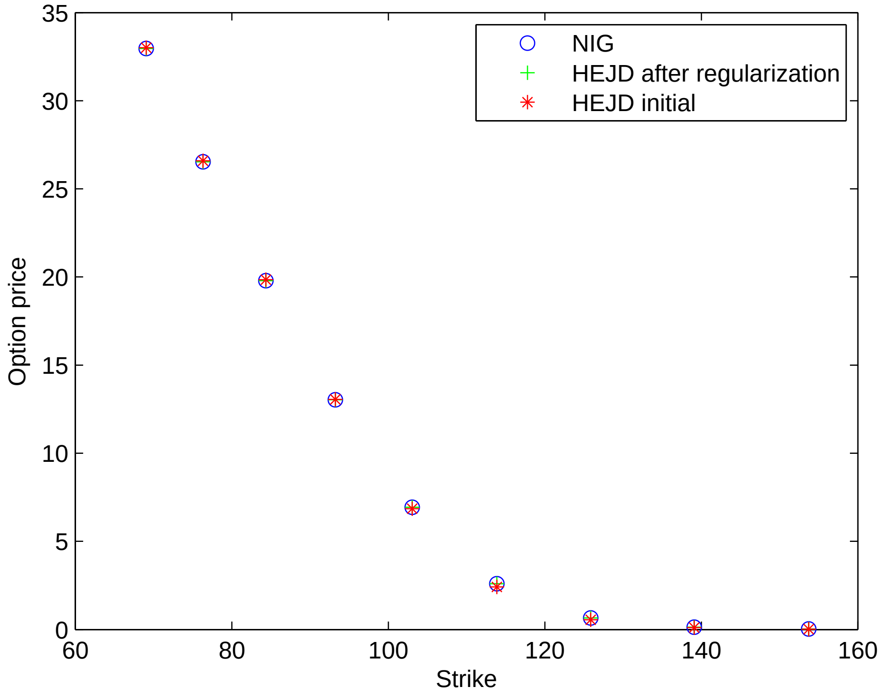


(b) Call prices

Figure 3: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ and $N = 14$.

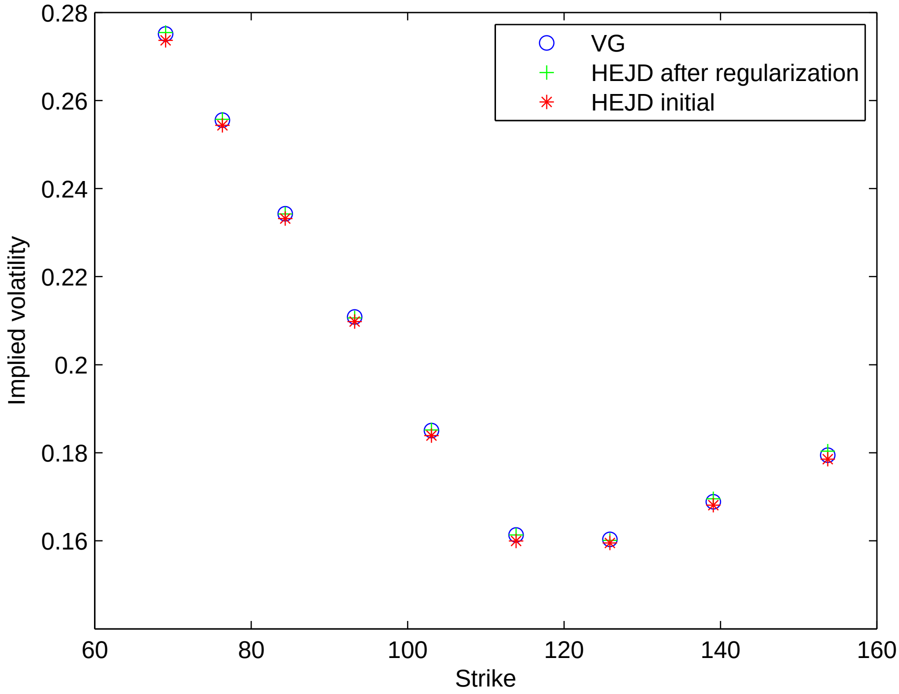


(a) Implied Volatility

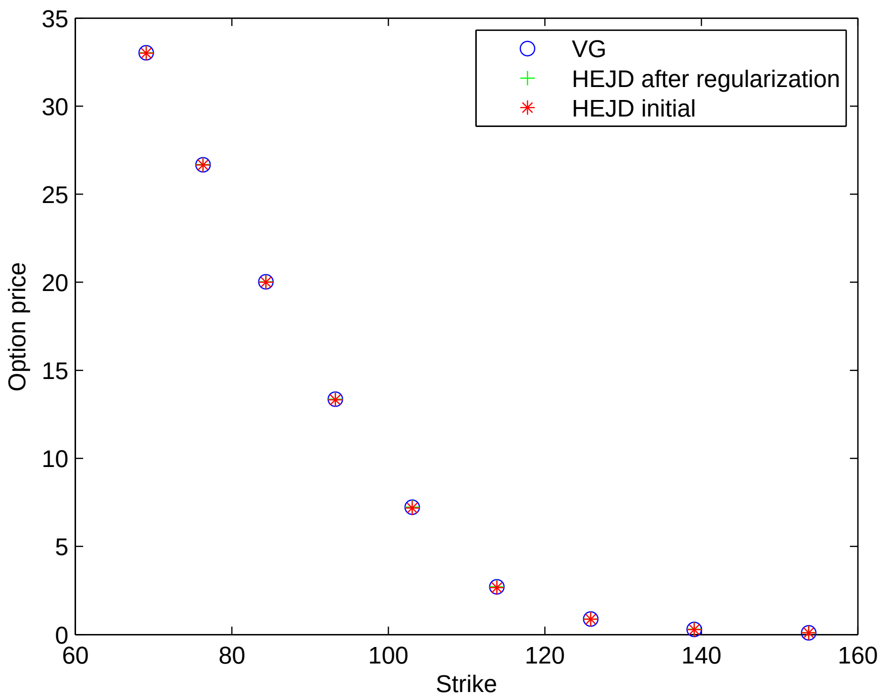


(b) Call prices

Figure 4: NIG results with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.



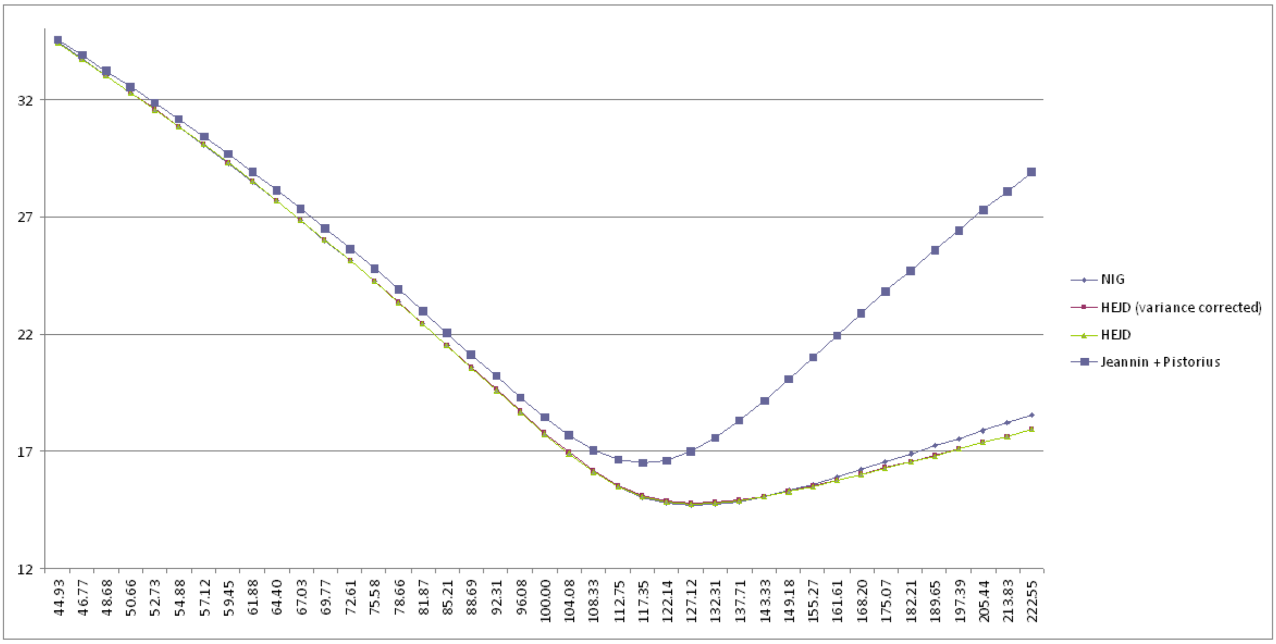
(a) Implied Volatility



(b) Call prices

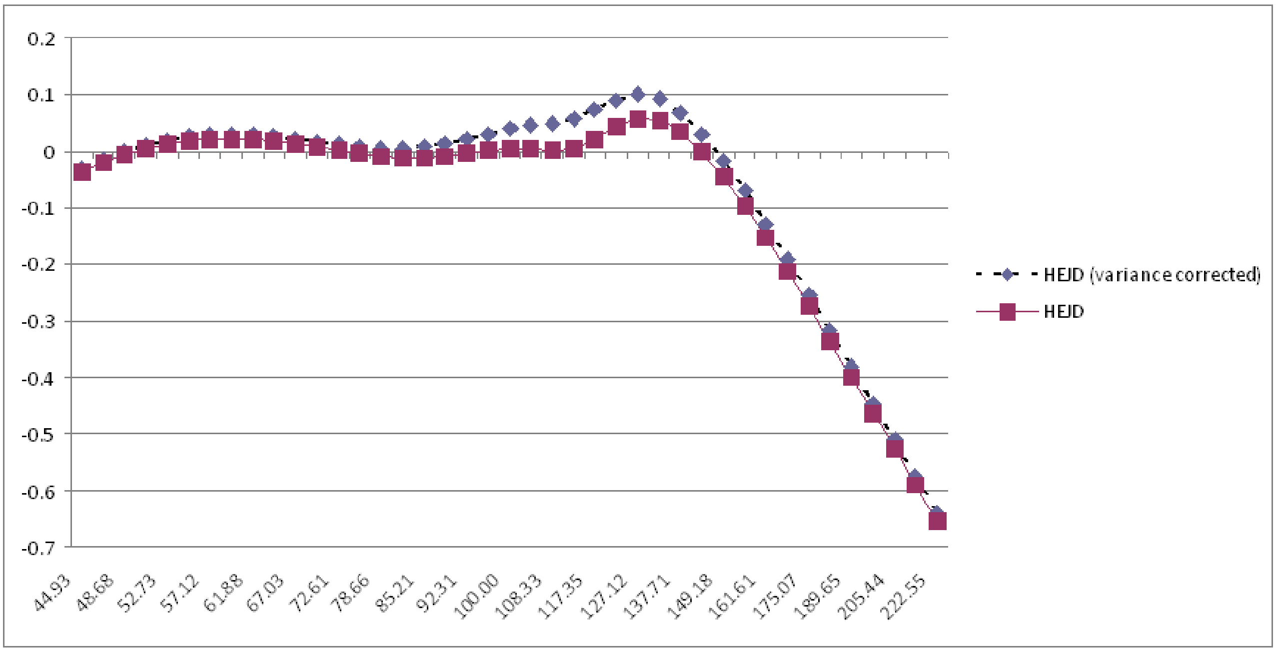
Figure 5: VG results with $C = 0.925$, $G = 4.667$, $M = 11.876$ and $N = 14$.

- Results for $S_{t_0} = 100$, $r = 0$, $q = 0$ and $T = 1$



$k =$

(a) Implied volatilities (expressed as a percentage)

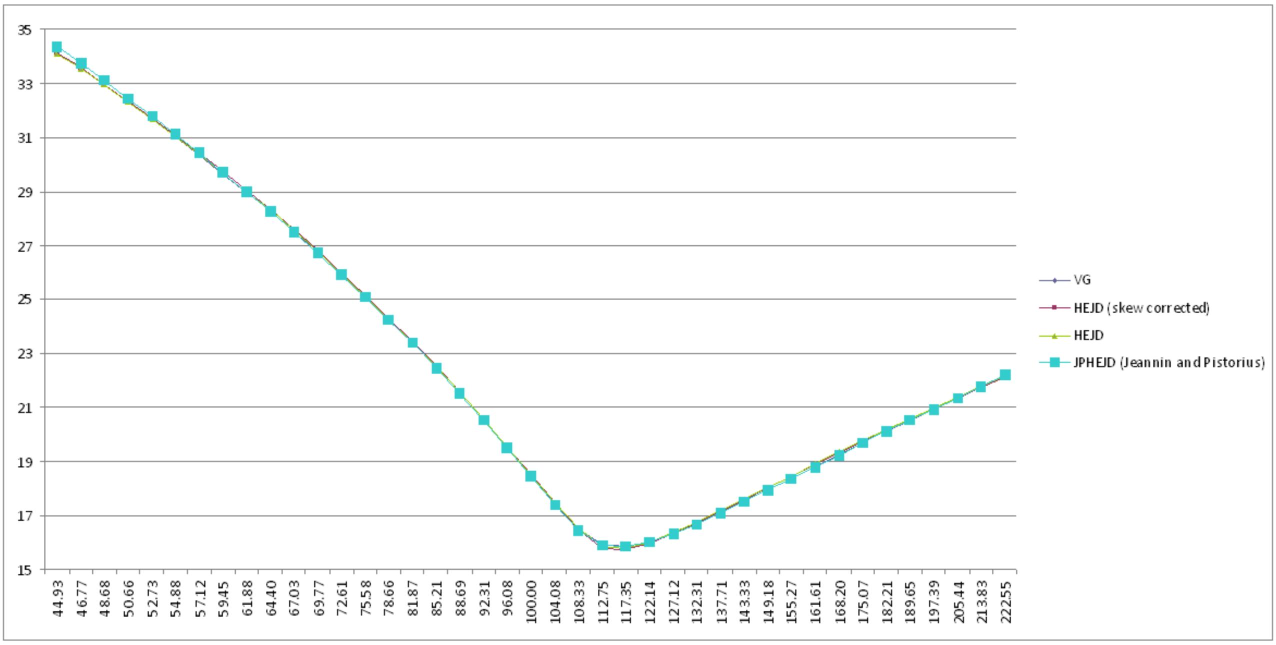


(b) Errors in implied volatilities (expressed as percentage points) compared to the original NIG process

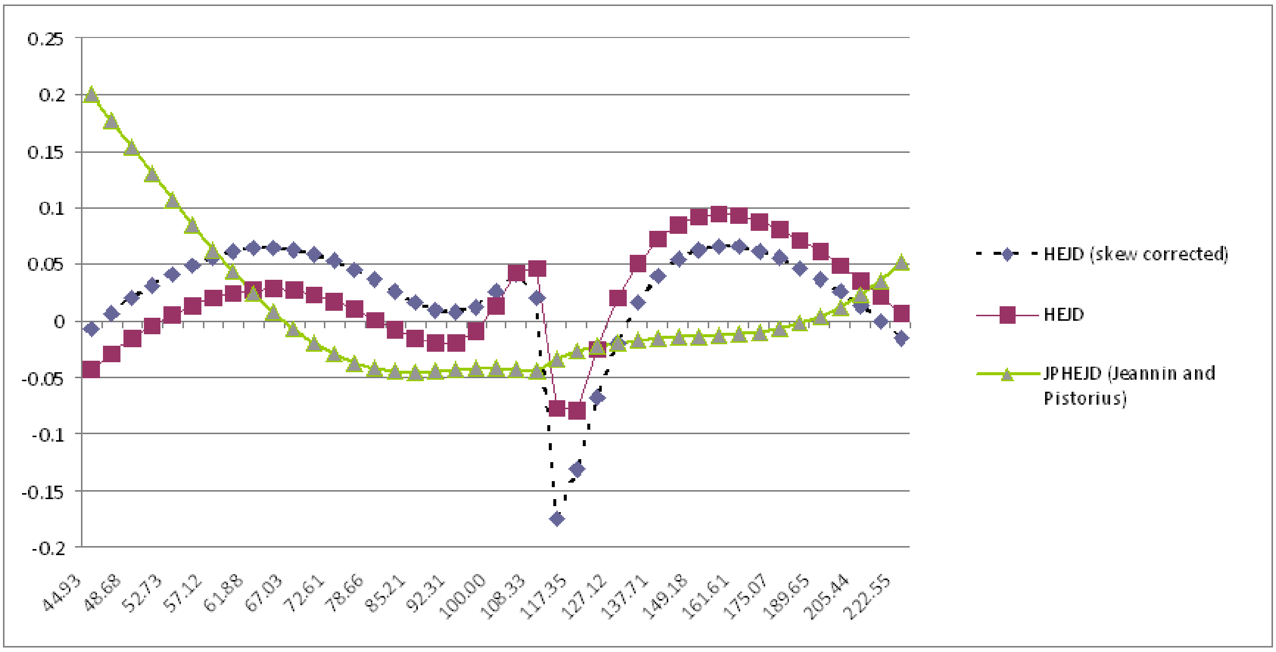
Figure 6: NIG results with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.

Strikes	Implied Volatilities			
	NIG	$HEJD_{\sigma^2-corr}$	$HEJD$	$HEJD_{Jeannin-Pistorius}$
44.932896	0.344551	0.344235	0.344180	0.345625
46.766643	0.337537	0.337390	0.337333	0.339128
48.675226	0.330407	0.330401	0.330342	0.332489
50.661699	0.323155	0.323264	0.323202	0.325699
52.729242	0.315776	0.315971	0.315907	0.318749
54.881164	0.308264	0.308519	0.308451	0.311633
57.120906	0.300611	0.300900	0.300829	0.304341
59.452055	0.292812	0.293111	0.293036	0.296866
61.878339	0.284859	0.285148	0.285067	0.289200
64.403642	0.276745	0.277005	0.276919	0.281335
67.032005	0.268464	0.268682	0.268590	0.273267
69.767633	0.260008	0.260178	0.260077	0.264990
72.614904	0.251373	0.251493	0.251383	0.256505
75.578374	0.242556	0.242633	0.242511	0.247815
78.662786	0.233557	0.233605	0.233469	0.238932
81.873075	0.224383	0.224426	0.224272	0.229878
85.214379	0.215053	0.215119	0.214943	0.220688
88.692044	0.205605	0.205725	0.205522	0.211428
92.311635	0.196103	0.196307	0.196069	0.202201
96.078944	0.186666	0.186967	0.186686	0.193176
100.000000	0.177485	0.177876	0.177540	0.184625
104.081077	0.168856	0.169305	0.168905	0.176959
108.328707	0.161189	0.161674	0.161208	0.170744
112.749685	0.154943	0.155505	0.154995	0.166615
117.351087	0.150476	0.151199	0.150687	0.165077
122.140276	0.147871	0.148779	0.148303	0.166306
127.124915	0.146931	0.147928	0.147499	0.170093
132.312981	0.147310	0.148230	0.147847	0.175975
137.712776	0.148659	0.149336	0.148994	0.183396
143.332941	0.150693	0.150995	0.150686	0.191832
149.182470	0.153198	0.153029	0.152748	0.200855
155.270722	0.156026	0.155318	0.155061	0.210154
161.607440	0.159070	0.157780	0.157542	0.219522
168.202765	0.162258	0.160356	0.160135	0.228828
175.067250	0.165539	0.163007	0.162800	0.237998
182.211880	0.168875	0.165703	0.165508	0.246992
189.648088	0.172242	0.168425	0.168240	0.255791
197.387773	0.175620	0.171157	0.170982	0.264389
205.443321	0.178997	0.173889	0.173722	0.272786
213.827622	0.182362	0.176613	0.176454	0.280987
222.554093	0.185709	0.179324	0.179172	0.289000

Table 15: Implied volatilities (expressed as decimals) for the NIG process and the approximating HEJD processes, with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.



(a) Implied volatilities (expressed as a percentage)



(b) Errors in implied volatilities (expressed as percentage points) compared to the original VG process

Figure 7: VG results with $C = 0.925$, $G = 4.667$, $M = 11.876$ and $N = 14$

Strikes	Implied Volatilities			
	VG	$HEJD_{skew-corr}$	$HEJD$	$HEJD_{Jeannin-Pistorius}$
44.932896	0.341684	0.341613	0.341257	0.343690
46.766643	0.335668	0.335736	0.335379	0.337436
48.675226	0.329534	0.329729	0.329372	0.331065
50.661699	0.323274	0.323584	0.323226	0.324571
52.729242	0.316880	0.317291	0.316932	0.317949
54.881164	0.310345	0.310839	0.310479	0.311192
57.120906	0.303657	0.304219	0.303858	0.304292
59.452055	0.296806	0.297416	0.297055	0.297241
61.878339	0.289780	0.290418	0.290057	0.290029
64.403642	0.282565	0.283210	0.282849	0.282646
67.032005	0.275146	0.275776	0.275414	0.275078
69.767633	0.267505	0.268097	0.267737	0.267310
72.614904	0.259622	0.260155	0.259797	0.259326
75.578374	0.251476	0.251930	0.251576	0.251103
78.662786	0.243041	0.243402	0.243055	0.242618
81.873075	0.234290	0.234551	0.234214	0.233842
85.214379	0.225193	0.225360	0.225039	0.224741
88.692044	0.215722	0.215818	0.215520	0.215280
92.311635	0.205856	0.205927	0.205665	0.205429
96.078944	0.195599	0.195718	0.195510	0.195181
100.000000	0.185030	0.185281	0.185160	0.184609
104.081077	0.174451	0.174856	0.174877	0.174016
108.328707	0.164863	0.165067	0.165330	0.164423
112.749685	0.159364	0.157615	0.158588	0.159023
117.351087	0.158655	0.157342	0.157864	0.158386
122.140276	0.160350	0.159667	0.160104	0.160126
127.124915	0.163297	0.163105	0.163494	0.163104
132.312981	0.166926	0.167081	0.167437	0.166755
137.712776	0.170931	0.171321	0.171652	0.170775
143.332941	0.175140	0.175680	0.175991	0.174994
149.182470	0.179450	0.180075	0.180370	0.179312
155.270722	0.183799	0.184458	0.184741	0.183669
161.607440	0.188150	0.188803	0.189075	0.188032
168.202765	0.192477	0.193093	0.193355	0.192379
175.067250	0.196766	0.197320	0.197573	0.196700
182.211880	0.201008	0.201477	0.201724	0.200986
189.648088	0.205195	0.205564	0.205804	0.205237
197.387773	0.209326	0.209580	0.209814	0.209451
205.443321	0.213399	0.213526	0.213755	0.213628
213.827622	0.217412	0.217404	0.217628	0.217770
222.554093	0.221367	0.221215	0.221435	0.221878

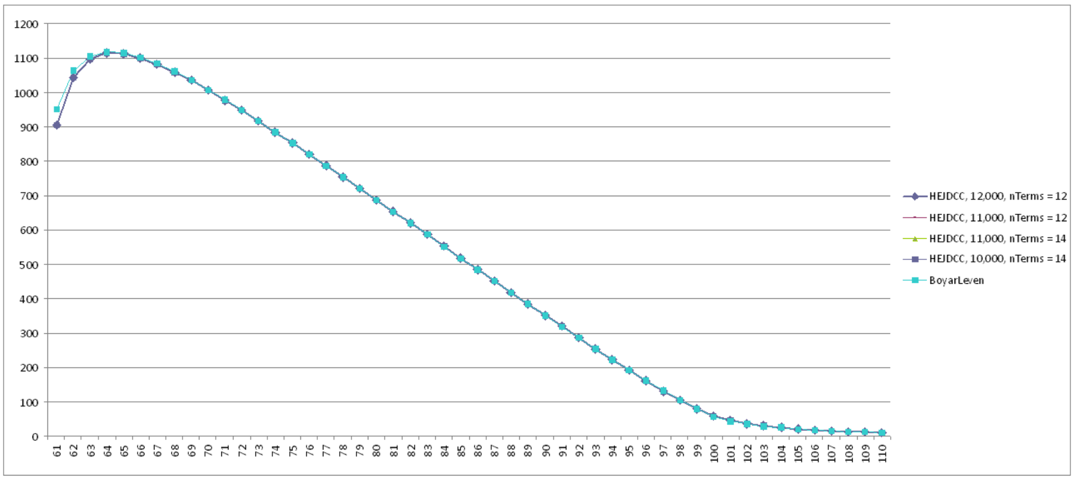
Table 16: Implied volatilities (expressed as decimals) for the VG process and the approximating HEJD processes with $C = 0.925$, $G = 4.667$, $M = 11.876$ and $N = 14$.

==*==*==*==*==*==*==*==*==*==*==*==*==*==*==*==*==*== Appendix

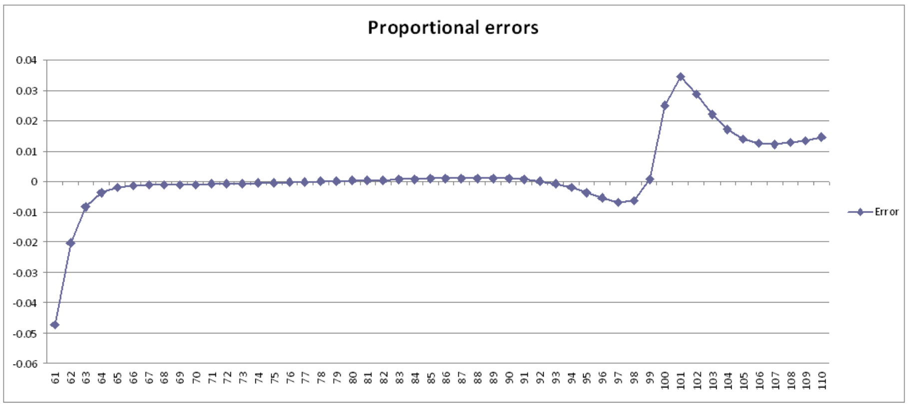
Appendix E : Barrier option price results

addcontentslinetocchapterAppendix E: Barrier option price results

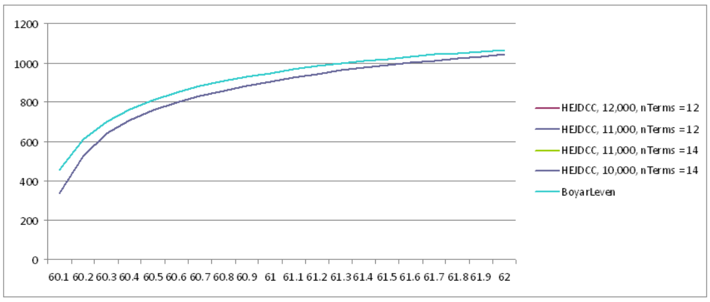
- Single barrier option price comparisons under CGMY



(a) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 61 to 110.



(b) Graph of proportional errors against spot, where the spot is expressed as a percentage of 3500, for percentages from 61 to 110.

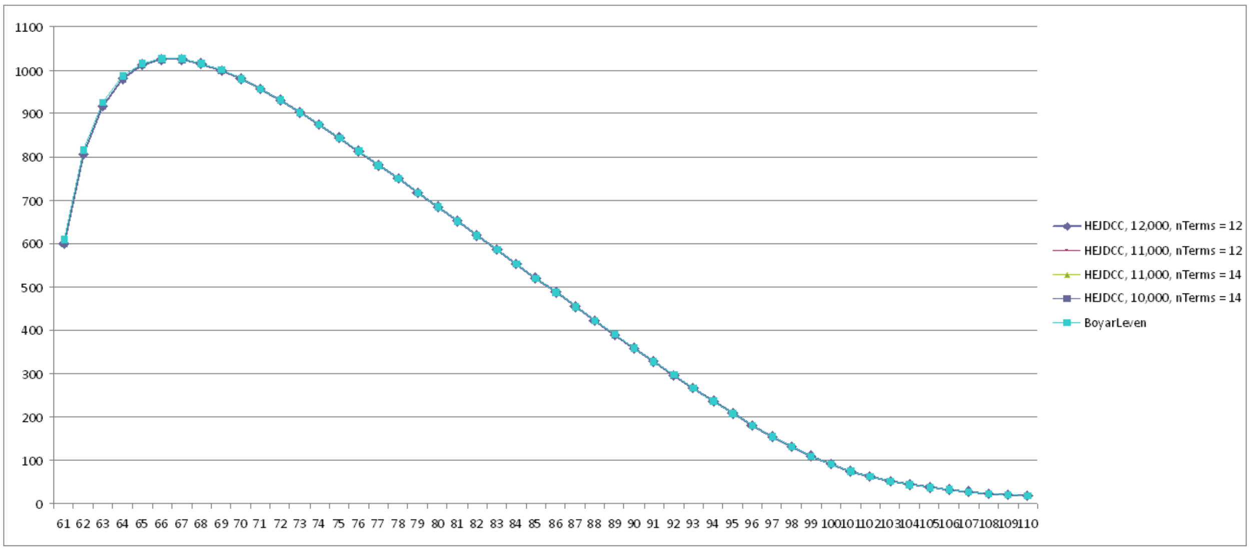


(c) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 60.1 to 62 ie the initial spot is close to the barrier.

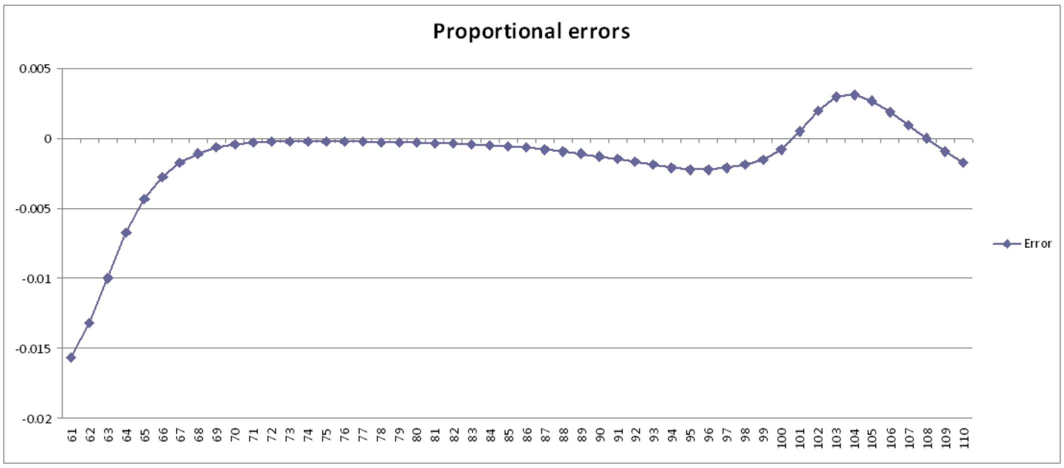
Figure 8: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.25$ and $N = 14$.

Spot as percentage of 3500	HEJDCC				BoyarLeven
	nTerms = 12		nTerms = 14		
	UB = 12000	UB = 11000	UB = 11000	UB = 10000	
61	905.5230	905.5230	905.4454	905.4454	950.4833
62	1042.6280	1042.6280	1042.6055	1042.6055	1064.2447
63	1096.6606	1096.6606	1096.6683	1096.6683	1106.0407
64	1114.1086	1114.1086	1114.1217	1114.1217	1118.3778
65	1112.8567	1112.8567	1112.8662	1112.8662	1115.1437
66	1101.0185	1101.0185	1101.0231	1101.0231	1102.5872
67	1082.6966	1082.6966	1082.6973	1082.6973	1084.0145
68	1060.1466	1060.1466	1060.1448	1060.1448	1061.3589
69	1034.6988	1034.6988	1034.6957	1034.6957	1035.8302
70	1007.1866	1007.1866	1007.1829	1007.1829	1008.2245
71	978.1593	978.1593	978.1555	978.1555	979.0854
72	947.9951	947.9951	947.9915	947.9915	948.7959
73	916.9636	916.9636	916.9603	916.9603	917.6328
74	885.2628	885.2628	885.2599	885.2599	885.7999
75	853.0420	853.0420	853.0395	853.0395	853.4509
76	820.4159	820.4159	820.4137	820.4137	820.7033
77	787.4747	787.4747	787.4729	787.4729	787.6487
78	754.2905	754.2905	754.2890	754.2890	754.3597
79	720.9220	720.9220	720.9208	720.9208	720.8950
80	687.4178	687.4178	687.4168	687.4168	687.3031
81	653.8190	653.8190	653.8182	653.8182	653.6251
82	620.1611	620.1611	620.1605	620.1605	619.8967
83	586.4755	586.4755	586.4751	586.4751	586.1499
84	552.7905	552.7905	552.7903	552.7903	552.4145
85	519.1329	519.1329	519.1328	519.1328	518.7191
86	485.5283	485.5283	485.5284	485.5284	485.0923
87	452.0026	452.0026	452.0028	452.0028	451.5641
88	418.5829	418.5829	418.5832	418.5832	418.1662
89	385.2993	385.2993	385.2996	385.2996	384.9343
90	352.1865	352.1865	352.1868	352.1868	351.9087
91	319.2871	319.2871	319.2874	319.2874	319.1372
92	286.6559	286.6559	286.6560	286.6560	286.6769
93	254.3663	254.3663	254.3662	254.3662	254.5987
94	222.5209	222.5209	222.5206	222.5206	222.9928
95	191.2672	191.2672	191.2668	191.2668	191.9777
96	160.8235	160.8235	160.8233	160.8233	161.7160
97	131.5201	131.5201	131.5205	131.5205	132.4430
98	103.8707	103.8707	103.8722	103.8722	104.5295
99	78.7026	78.7026	78.7051	78.7051	78.6479
100	57.9328	57.9328	57.9333	57.9333	56.5216
101	44.9917	44.9917	44.9944	44.9944	43.4931
102	36.3823	36.3823	36.3840	36.3840	35.3640
103	30.0588	30.0588	30.0594	30.0594	29.4112
104	25.2337	25.2336	25.2336	25.2336	24.8122
105	21.4441	21.4441	21.4438	21.4438	21.1495
106	18.3995	18.3995	18.3993	18.3993	18.1728
107	15.9087	15.9087	15.9086	15.9086	15.7174
108	13.8409	13.8409	13.8410	13.8410	13.6685
109	12.1037	12.1037	12.1040	12.1040	11.9429
110	10.6300	10.6300	10.6304	10.6304	10.4784

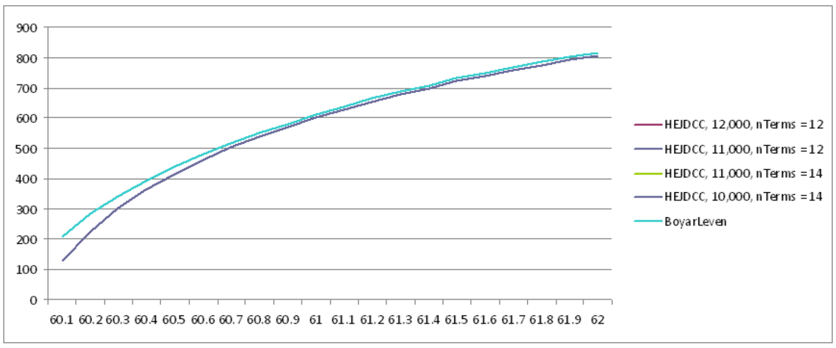
Table 17: CGMY down-and-out put options prices with $C = 1$, $G = 9$, $M = 8$, $Y = 0.25$ and $N = 14$.



(a) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 61 to 110.



(b) Graph of proportional errors against spot, where the spot is expressed as a percentage of 3500, for percentages from 61 to 110.

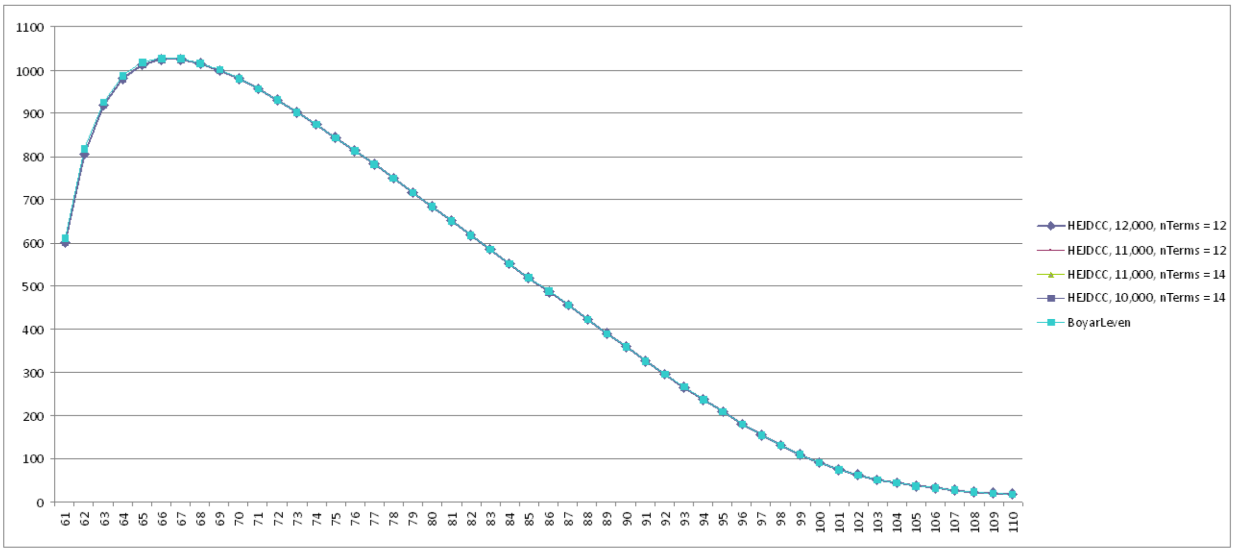


(c) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 60.1 to 62 ie the initial spot is close to the barrier.

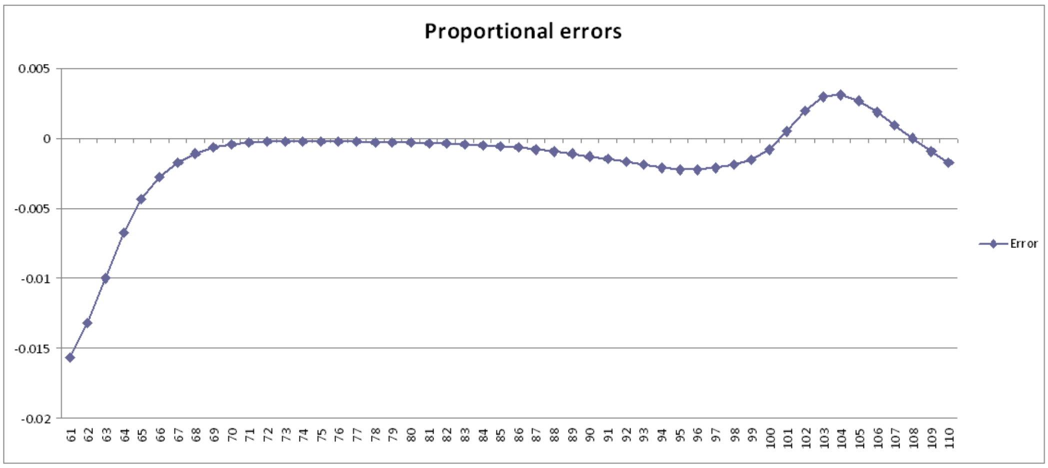
Figure 9: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and $N = 14$.

Spot as percentage of 3500	HEJDCC				BoyarLeven
	nTerms = 12		nTerms = 14		
	UB = 12000	UB = 11000	UB = 11000	UB = 10000	
61	601.3806	601.3806	601.2757	601.2757	610.9543
62	806.1518	806.1518	806.1751	806.1751	816.9377
63	917.8685	917.8685	917.8973	917.8973	927.1703
64	980.0878	980.0878	980.1015	980.1015	986.7716
65	1012.1249	1012.1249	1012.1261	1012.1261	1016.5752
66	1024.7589	1024.7589	1024.7540	1024.7540	1027.6004
67	1024.4365	1024.4365	1024.4301	1024.4301	1026.2029
68	1015.2020	1015.2020	1015.1966	1015.1966	1016.2829
69	999.6878	999.6878	999.6841	999.6841	1000.3486
70	979.6594	979.6594	979.6575	979.6575	980.0735
71	956.3310	956.3310	956.3305	956.3305	956.6078
72	930.5560	930.5560	930.5564	930.5564	930.7624
73	902.9457	902.9457	902.9467	902.9467	903.1211
74	873.9457	873.9457	873.9470	873.9470	874.1121
75	843.8864	843.8864	843.8877	843.8877	844.0550
76	813.0161	813.0161	813.0174	813.0174	813.1924
77	781.5253	781.5253	781.5264	781.5264	781.7114
78	749.5618	749.5618	749.5627	749.5627	749.7583
79	717.2430	717.2430	717.2437	717.2437	717.4504
80	684.6639	684.6639	684.6644	684.6644	684.8827
81	651.9035	651.9035	651.9039	651.9039	652.1349
82	619.0292	619.0292	619.0294	619.0294	619.2750
83	586.1004	586.1004	586.1005	586.1005	586.3630
84	553.1718	553.1718	553.1718	553.1718	553.4542
85	520.2952	520.2952	520.2952	520.2952	520.6007
86	487.5224	487.5224	487.5223	487.5223	487.8543
87	454.9064	454.9064	454.9064	454.9064	455.2677
88	422.5044	422.5044	422.5044	422.5044	422.8972
89	390.3792	390.3792	390.3792	390.3792	390.8043
90	358.6022	358.6022	358.6021	358.6021	359.0583
91	327.2559	327.2559	327.2559	327.2559	327.7389
92	296.4382	296.4382	296.4381	296.4381	296.9402
93	266.2661	266.2661	266.2659	266.2659	266.7748
94	236.8821	236.8821	236.8817	236.8817	237.3799
95	208.4609	208.4609	208.4603	208.4603	208.9258
96	181.2195	181.2195	181.2187	181.2187	181.6272
97	155.4287	155.4287	155.4280	155.4280	155.7585
98	131.4307	131.4307	131.4304	131.4304	131.6762
99	109.6742	109.6742	109.6745	109.6745	109.8431
100	90.7561	90.7561	90.7563	90.7563	90.8289
101	75.1560	75.1560	75.1568	75.1568	75.1193
102	62.6912	62.6912	62.6907	62.6907	62.5689
103	52.7063	52.7063	52.7050	52.7050	52.5510
104	44.6162	44.6162	44.6150	44.6150	44.4766
105	38.0010	38.0010	38.0002	38.0002	37.8984
106	32.5481	32.5481	32.5478	32.5478	32.4858
107	28.0203	28.0203	28.0203	28.0203	27.9931
108	24.2350	24.2350	24.2352	24.2352	24.2352
109	21.0509	21.0509	21.0512	21.0512	21.0705
110	18.3572	18.3572	18.3575	18.3575	18.3893

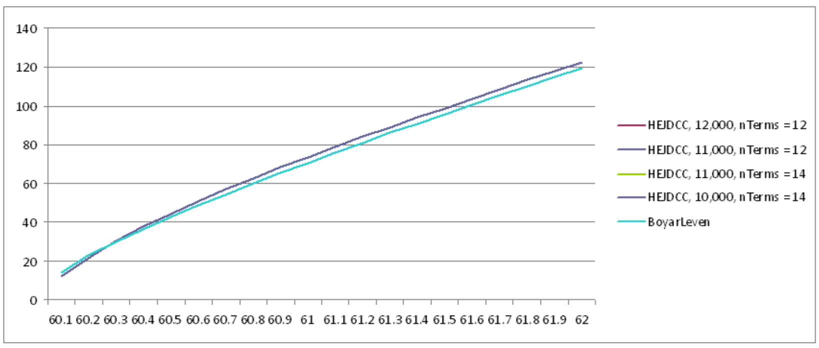
Table 18: CGMY down-and-out put options prices with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and $N = 14$.



(a) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 61 to 110.



(b) Graph of proportional errors against spot, where the spot is expressed as a percentage of 3500, for percentages from 61 to 110.



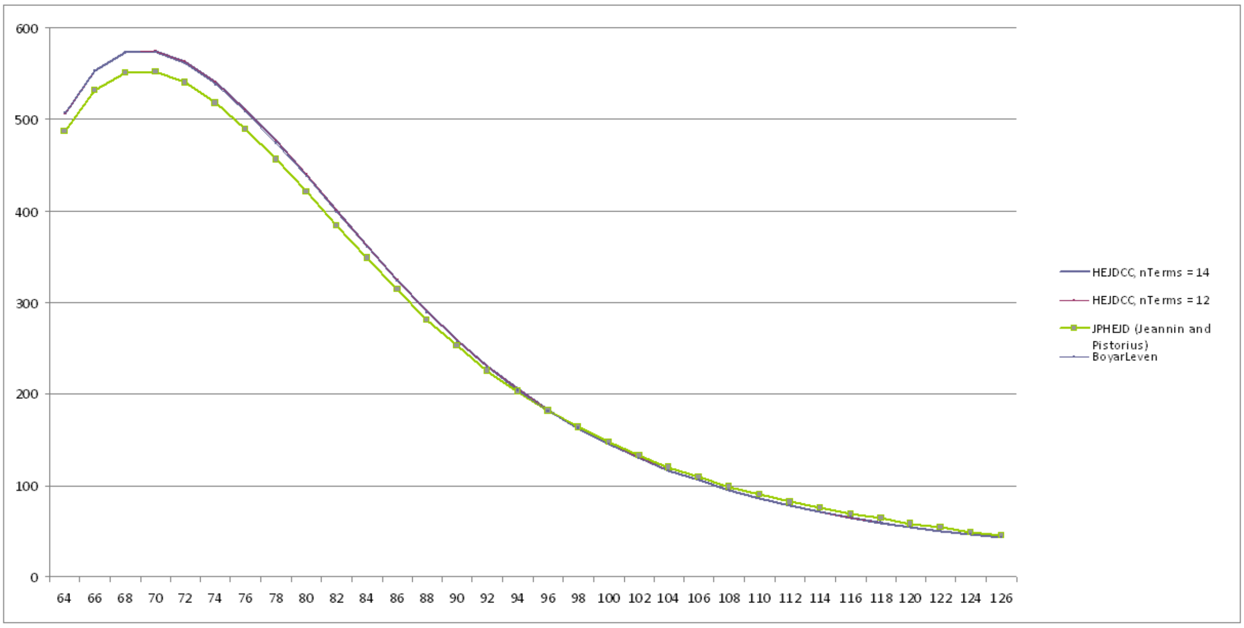
(c) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 60.1 to 62 ie the initial spot is close to the barrier.

Figure 10: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ and $N = 14$.

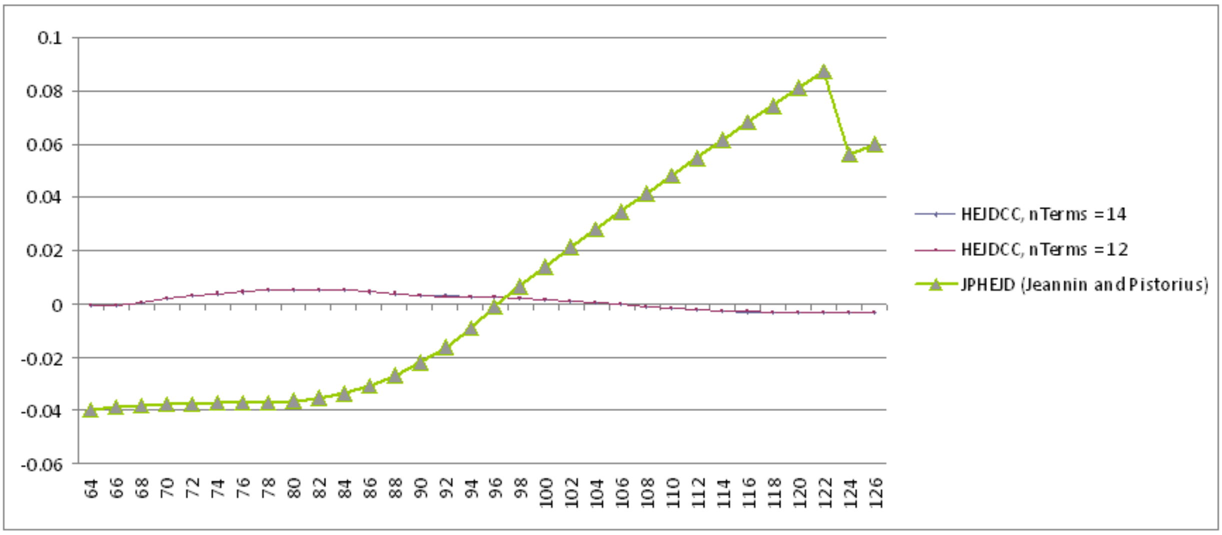
Spot as percentage of 3500	HEJDCC				BoyarLeven
	nTerms = 12		nTerms = 14		
	UB = 12000	UB = 11000	UB = 11000	UB = 10000	
61	73.5585	73.5585	73.5581	73.5581	70.6425
62	122.4994	122.4994	122.4950	122.4950	119.6255
63	166.8811	166.8811	166.8752	166.8752	164.3627
64	208.2931	208.2931	208.2933	208.2933	206.1131
65	246.9967	246.9967	247.0084	247.0084	245.1009
66	282.9659	282.9659	282.9886	282.9886	281.2973
67	316.1058	316.1058	316.1339	316.1339	314.6101
68	346.3203	346.3203	346.3460	346.3460	344.9475
69	373.5378	373.5378	373.5533	373.5533	372.2430
70	397.7196	397.7196	397.7193	397.7193	396.4639
71	418.8624	418.8624	418.8435	418.8435	417.6141
72	436.9972	436.9972	436.9600	436.9600	435.7319
73	452.1869	452.1869	452.1343	452.1343	450.8874
74	464.5227	464.5227	464.4591	464.4591	463.1780
75	474.1194	474.1194	474.0503	474.0503	472.7241
76	481.1118	481.1118	481.0429	481.0429	479.6650
77	485.6499	485.6499	485.5865	485.5865	484.1537
78	487.8953	487.8953	487.8416	487.8416	486.3540
79	488.0167	488.0167	487.9760	487.9760	486.4360
80	486.1871	486.1871	486.1617	486.1617	484.5734
81	482.5809	482.5809	482.5717	482.5717	480.9406
82	477.3708	477.3708	477.3778	477.3778	475.7102
83	470.7263	470.7263	470.7485	470.7485	469.0512
84	462.8116	462.8116	462.8475	462.8475	461.1274
85	453.7843	453.7843	453.8318	453.8318	452.0959
86	443.7948	443.7948	443.8515	443.8515	442.1065
87	432.9849	432.9849	433.0483	433.0483	431.3007
88	421.4876	421.4876	421.5552	421.5552	419.8111
89	409.4272	409.4272	409.4967	409.4967	407.7618
90	396.9185	396.9185	396.9877	396.9877	395.2674
91	384.0674	384.0674	384.1344	384.1344	382.4338
92	370.9707	370.9707	371.0340	371.0340	369.3575
93	357.7166	357.7166	357.7749	357.7749	356.1268
94	344.3848	344.3848	344.4371	344.4371	342.8213
95	331.0469	331.0469	331.0925	331.0925	329.5125
96	317.7669	317.7669	317.8052	317.8052	316.2644
97	304.6015	304.6015	304.6322	304.6322	303.1334
98	291.6004	291.6004	291.6234	291.6234	290.1694
99	278.8071	278.8071	278.8225	278.8225	277.4157
100	266.2590	266.2590	266.2670	266.2670	264.9096
101	253.9881	253.9881	253.9890	253.9890	252.6829
102	242.0211	242.0211	242.0155	242.0155	240.7622
103	230.3804	230.3804	230.3688	230.3688	229.1697
104	219.0838	219.0838	219.0667	219.0667	217.9230
105	208.1454	208.1454	208.1235	208.1235	207.0358
106	197.5757	197.5757	197.5495	197.5495	196.5185
107	187.3820	187.3820	187.3523	187.3523	186.3781
108	177.5687	177.5687	177.5361	177.5361	176.6189
109	168.1377	168.1377	168.1031	168.1031	167.2425
110	159.0888	159.0888	159.0527	159.0527	158.2484

Table 19: CGMY down-and-out put option prices with $C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ and $N = 14$.

- Single barrier option price comparisons under NIG



(a) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 64 to 126.



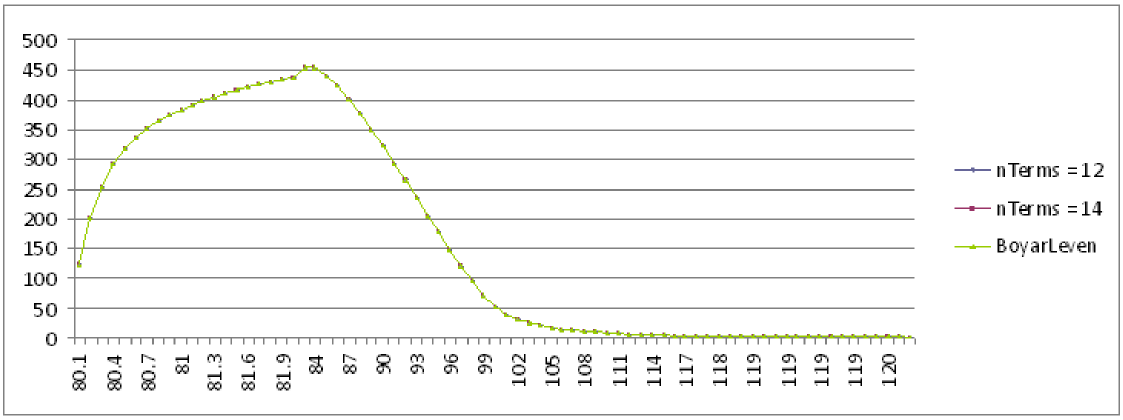
(b) Graph of proportional errors against spot, where the spot is expressed as a percentage of 3500, for percentages from 64 to 126.

Figure 11: NIG results with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.

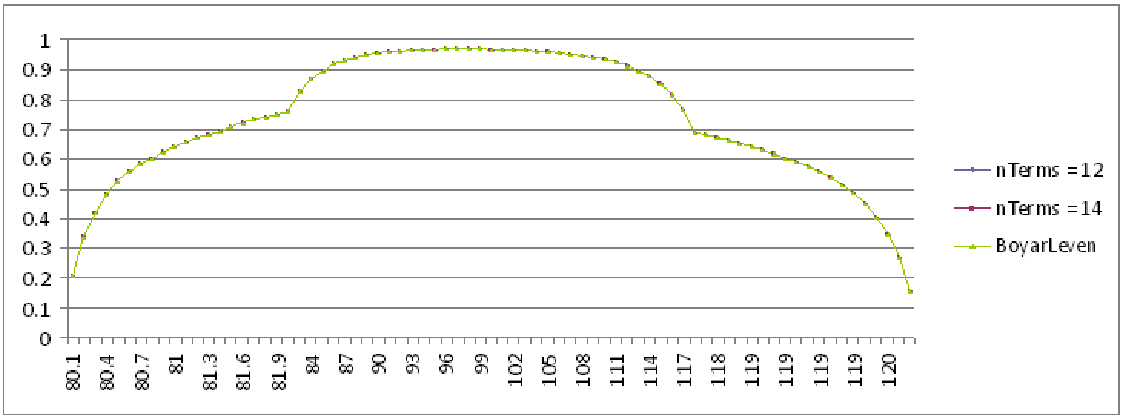
Spot as percentage of 3500	HEJDCC		JPHEJD	BoyarLeven
	nTerms = 14	nTerms = 12		
64	506.7577	506.7804	486.8291	507.0212
66	553.8066	553.8236	532.6638	554.2226
68	574.0845	574.0842	551.8006	573.7149
70	575.4591	575.4368	552.6467	574.2871
72	563.1071	563.0662	540.3645	561.3327
74	540.7391	540.6906	518.5689	538.5875
76	511.1880	511.1477	490.0120	508.8566
78	476.7393	476.7230	456.9306	474.3919
80	439.3267	439.3432	421.2329	437.1020
82	400.6369	400.6818	384.5789	398.6511
84	362.1431	362.1971	348.3821	360.4770
86	325.0816	325.1178	313.7675	323.7562
88	290.3910	290.3913	281.5287	289.3576
90	258.6547	258.6262	252.1222	257.8240
92	230.0973	230.0693	225.7084	229.3982
94	204.6714	204.6659	202.2236	204.0858
96	182.1974	182.2038	181.4602	181.7321
98	162.4416	162.4437	163.1318	162.0915
100	145.1212	145.1218	146.9159	144.8760
102	129.9477	129.9483	132.5359	129.8093
104	116.6529	116.6534	119.8498	116.6071
106	104.9927	104.9931	108.6566	105.0261
108	94.7501	94.7503	98.7645	94.8474
110	85.7348	85.7351	90.0030	85.8807
112	77.7819	77.7823	82.2233	77.9615
114	70.7490	70.7495	75.2968	70.9488
116	64.5136	64.5143	69.1130	64.7221
118	58.9708	58.9716	63.5768	59.1783
120	54.0306	54.0315	58.6068	54.2293
122	49.6160	49.6168	54.1331	49.7997
124	45.6606	45.6614	48.3773	45.8249
126	42.1076	42.1084	44.7756	42.2494

Table 20: NIG down-and-out put option prices with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.

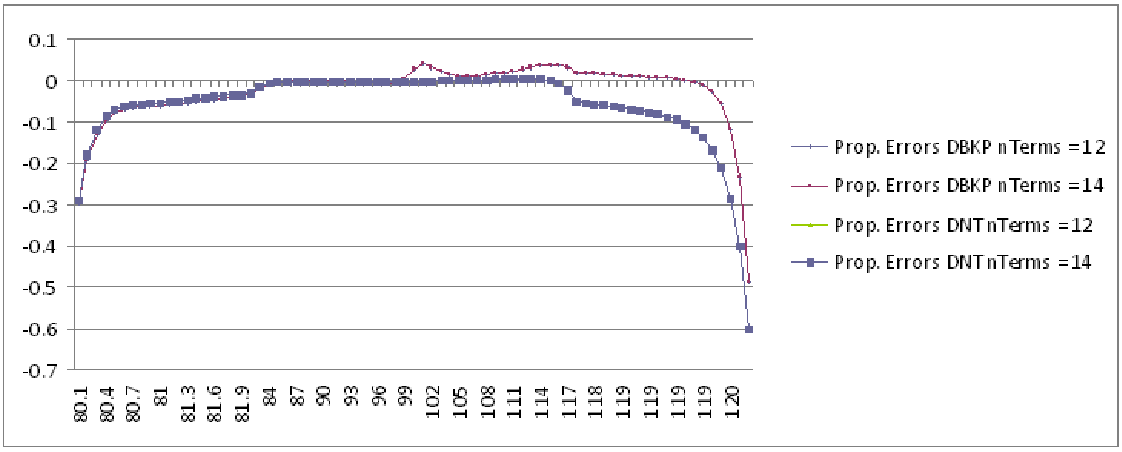
- Double barrier option price comparisons under CGMY



(a) Double barrier knockout put option prices



(b) DNT option prices

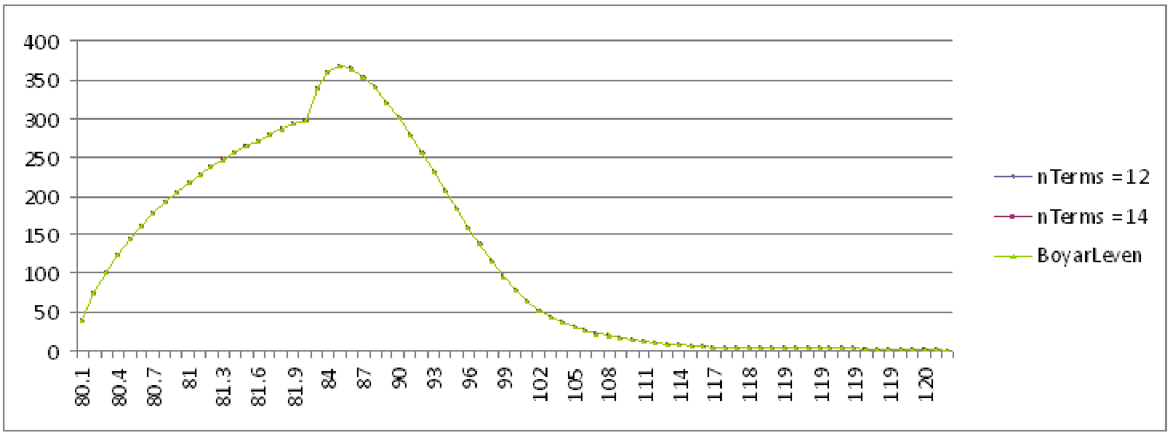


(c) Proportional errors for both double barrier knockout put (DBKP) option prices and double-no-touch (DNT) option prices

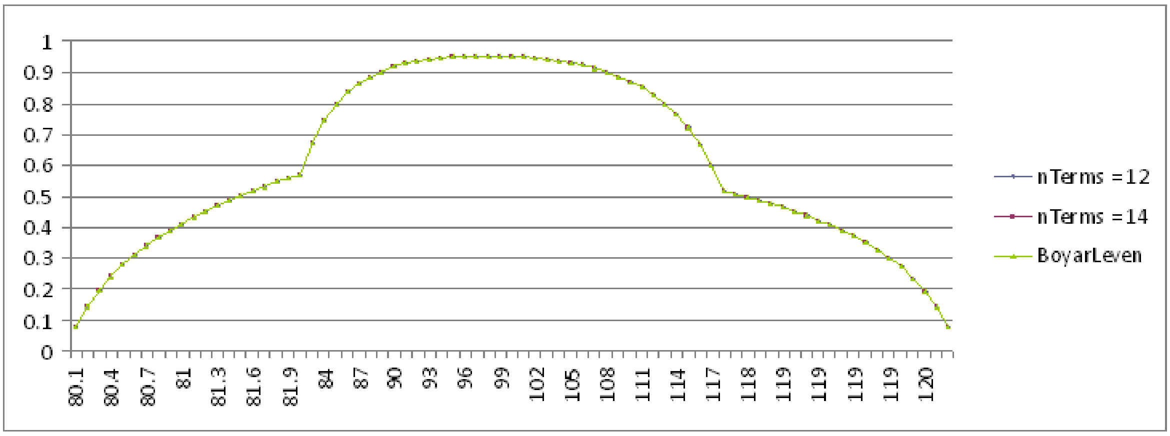
Figure 12: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.25$ and $N = 14$.

Spot as percentage of 3500	Double barrier knockout put options			DNT options		
	HEJDCC		BoyarLeven	HEJDCC		BoyarLeven
	nTerms = 12	nTerms = 14		nTerms = 12	nTerms = 14	
80.1	122.2533	122.2534	174.0610	0.2064	0.2064	0.2912
80.2	201.5358	201.5458	249.1757	0.3368	0.3368	0.4105
80.3	254.8077	254.8050	292.8343	0.4230	0.4230	0.4797
80.4	291.5694	291.5347	322.2530	0.4824	0.4823	0.5273
80.5	317.5670	317.5299	344.2066	0.5248	0.5248	0.5636
80.6	336.6114	336.5967	361.6260	0.5567	0.5567	0.5931
80.7	351.2913	351.2968	375.9142	0.5820	0.5821	0.6180
80.8	363.2645	363.2778	387.9048	0.6034	0.6034	0.6395
80.9	373.5027	373.5157	398.1158	0.6222	0.6223	0.6584
81	382.5353	382.5456	406.9096	0.6393	0.6394	0.6752
81.1	390.6415	390.6496	414.5405	0.6551	0.6551	0.6903
81.2	397.9743	397.9812	421.2012	0.6698	0.6698	0.7041
81.3	404.6273	404.6336	427.0373	0.6835	0.6835	0.7166
81.4	410.6670	410.6730	432.1642	0.6963	0.6963	0.7282
81.5	416.1467	416.1523	436.6738	0.7083	0.7084	0.7388
81.6	421.1121	421.1173	440.6411	0.7197	0.7197	0.7487
81.7	425.6036	425.6085	444.1283	0.7303	0.7303	0.7579
81.8	429.6578	429.6622	447.1866	0.7403	0.7403	0.7665
81.9	433.3074	433.3114	449.8604	0.7498	0.7498	0.7746
82	436.5822	436.5858	452.1865	0.7587	0.7587	0.7822
83	453.6285	453.6289	461.7749	0.8262	0.8262	0.8391
84	452.1946	452.1938	456.1509	0.8682	0.8682	0.8752
85	440.4243	440.4235	442.1934	0.8961	0.8961	0.9001
86	422.4263	422.4259	423.0977	0.9157	0.9157	0.9181
87	400.4696	400.4695	400.6149	0.9298	0.9298	0.9316
88	375.8951	375.8952	375.8113	0.9404	0.9404	0.9419
89	349.5430	349.5432	349.3872	0.9484	0.9484	0.9498
90	321.9702	321.9704	321.8322	0.9545	0.9545	0.9559
91	293.5692	293.5694	293.5088	0.9593	0.9593	0.9607
92	264.6374	264.6376	264.7016	0.9630	0.9630	0.9645
93	235.4219	235.4221	235.6499	0.9658	0.9658	0.9673
94	206.1530	206.1531	206.5710	0.9679	0.9679	0.9694
95	177.0740	177.0742	177.6811	0.9694	0.9694	0.9708
96	148.4768	148.4769	149.2196	0.9703	0.9703	0.9718
97	120.7493	120.7493	121.4836	0.9708	0.9708	0.9722
98	94.4507	94.4505	94.8960	0.9708	0.9708	0.9721
99	70.4449	70.4446	70.1807	0.9704	0.9704	0.9716
100	50.6785	50.6786	49.1722	0.9696	0.9696	0.9707
101	38.6062	38.6058	37.1513	0.9684	0.9684	0.9694
102	30.7510	30.7508	29.7947	0.9668	0.9668	0.9675
103	25.0836	25.0836	24.4947	0.9648	0.9648	0.9652
104	20.8306	20.8306	20.4562	0.9624	0.9624	0.9623
105	17.5407	17.5407	17.2785	0.9595	0.9595	0.9589
106	14.9332	14.9331	14.7233	0.9559	0.9559	0.9547
107	12.8251	12.8251	12.6353	0.9518	0.9518	0.9498
108	11.0927	11.0927	10.9076	0.9468	0.9468	0.9439
109	9.6496	9.6495	9.4630	0.9407	0.9407	0.9369
110	8.4333	8.4333	8.2445	0.9334	0.9334	0.9287
111	7.3980	7.3979	7.2086	0.9243	0.9243	0.9188
112	6.5087	6.5086	6.3215	0.9127	0.9127	0.9068
113	5.7378	5.7377	5.5565	0.8979	0.8979	0.8923
114	5.0632	5.0631	4.8917	0.8783	0.8783	0.8744
115	4.4657	4.4656	4.3089	0.8520	0.8520	0.8518
116	3.9274	3.9274	3.7917	0.8157	0.8157	0.8225
117	3.4296	3.4295	3.3238	0.7644	0.7644	0.7829
118	2.9481	2.9480	2.8839	0.6897	0.6897	0.7254
118.1	2.8996	2.8995	2.8401	0.6804	0.6804	0.7181
118.2	2.8508	2.8508	2.7962	0.6708	0.6708	0.7104
118.3	2.8018	2.8018	2.7520	0.6607	0.6607	0.7023
118.4	2.7524	2.7524	2.7074	0.6502	0.6502	0.6937
118.5	2.7026	2.7026	2.6624	0.6392	0.6392	0.6845
118.6	2.6523	2.6523	2.6168	0.6276	0.6276	0.6749
118.7	2.6014	2.6014	2.5705	0.6155	0.6155	0.6645
118.8	2.5498	2.5497	2.5232	0.6026	0.6026	0.6534
118.9	2.4971	2.4971	2.4747	0.5887	0.5887	0.6415
119	2.4429	2.4429	2.4247	0.5736	0.5736	0.6285
119.1	2.3864	2.3864	2.3727	0.5566	0.5566	0.6144
119.2	2.3260	2.3260	2.3183	0.5372	0.5371	0.5989
119.3	2.2588	2.2588	2.2607	0.5141	0.5141	0.5817
119.4	2.1796	2.1796	2.1987	0.4859	0.4859	0.5624
119.5	2.0790	2.0790	2.1306	0.4506	0.4506	0.5403
119.6	1.9405	1.9404	2.0537	0.4051	0.4051	0.5145
119.7	1.7348	1.7348	1.9623	0.3454	0.3454	0.4830
119.8	1.4115	1.4115	1.8452	0.2652	0.2652	0.4425
119.9	0.8833	0.8832	1.7161	0.1549	0.1549	0.3905

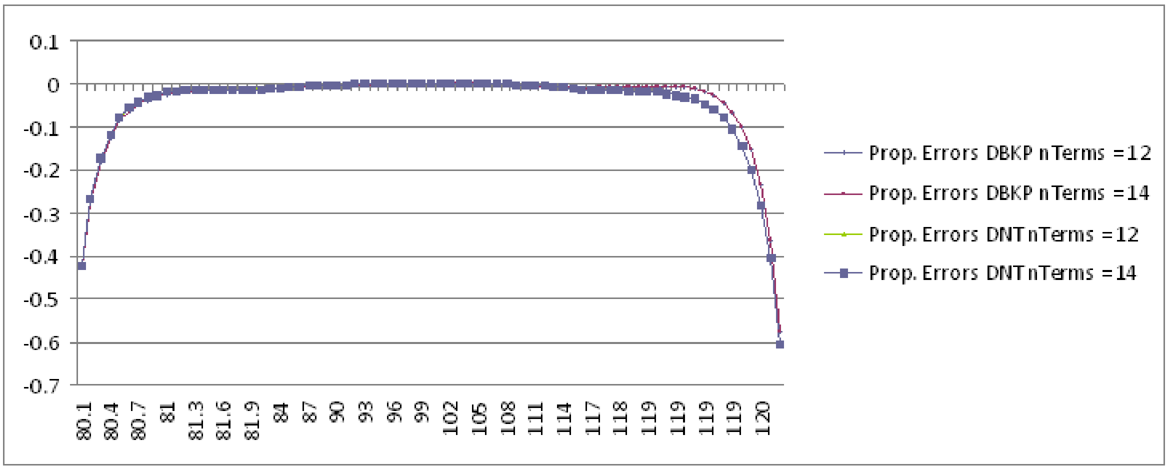
Table 21: CGMY double barrier option prices with $C = 1$, $G = 9$, $M = 8$, $Y = 0.25$ and $N = 14$.



(a) Double barrier knockout put option prices



(b) DNT option prices

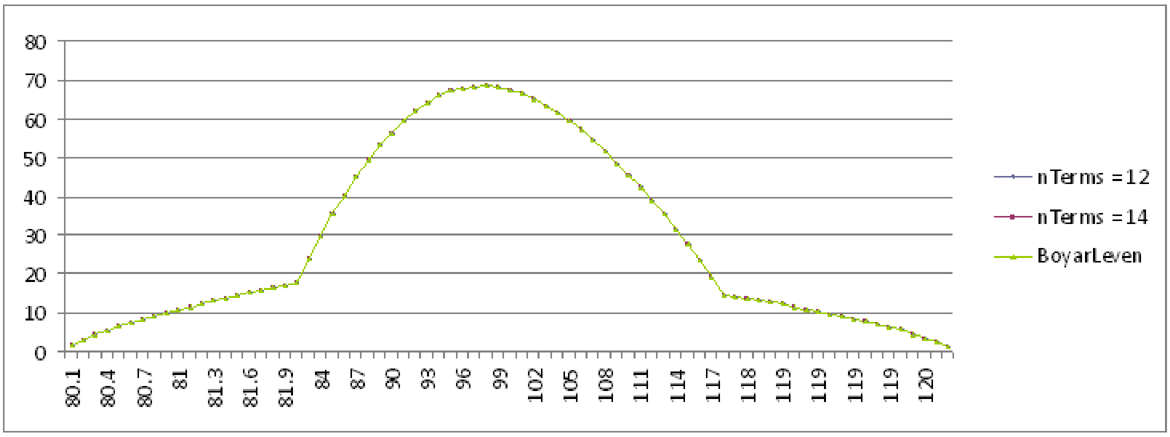


(c) Proportional errors for both double barrier knockout put (DBKP) option prices and double-no-touch (DNT) option prices

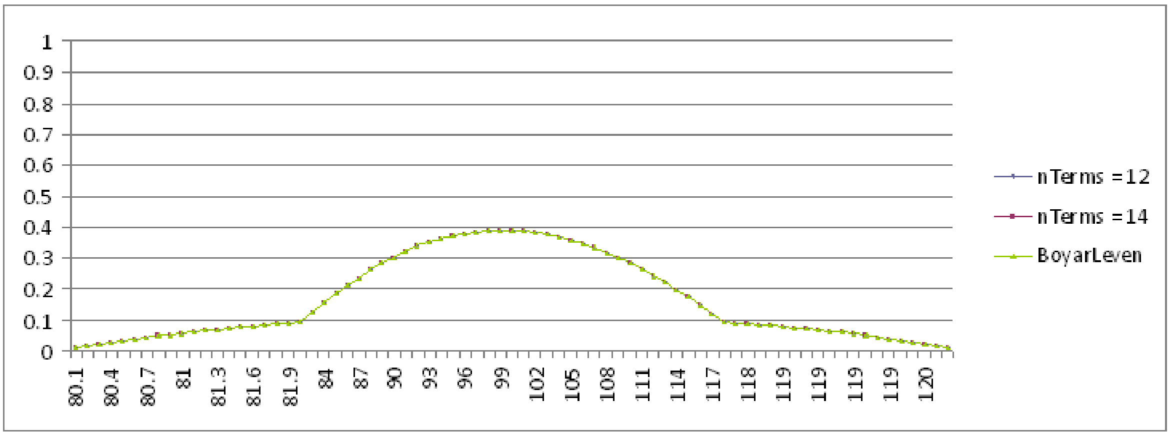
Figure 13: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and $N = 14$.

Spot as percentage of 3500	Double barrier knockout put options			DNT options		
	HEJDCC		BoyarLeven	HEJDCC		BoyarLeven
	nTerms = 12	nTerms = 14		nTerms = 12	nTerms = 14	
80.1	40.3367	40.3371	70.5490	0.0790	0.0790	0.1370
80.2	73.2870	73.2876	100.8295	0.1426	0.1426	0.1942
80.3	100.7504	100.7530	122.9329	0.1948	0.1948	0.2353
80.4	124.0876	124.0966	141.5406	0.2387	0.2387	0.2697
80.5	144.2777	144.2957	158.0171	0.2764	0.2764	0.3000
80.6	162.0293	162.0533	172.9799	0.3092	0.3093	0.3275
80.7	177.8559	177.8781	186.7462	0.3385	0.3385	0.3529
80.8	192.1287	192.1406	199.5129	0.3648	0.3649	0.3765
80.9	205.1166	205.1123	211.4078	0.3889	0.3889	0.3986
81	217.0158	216.9938	222.5256	0.4110	0.4110	0.4194
81.1	227.9729	227.9354	232.9376	0.4316	0.4315	0.4391
81.2	238.1009	238.0527	242.7025	0.4507	0.4507	0.4577
81.3	247.4901	247.4369	251.8688	0.4687	0.4687	0.4753
81.4	256.2158	256.1627	260.4789	0.4857	0.4856	0.4921
81.5	264.3417	264.2927	268.5697	0.5017	0.5016	0.5081
81.6	271.9233	271.8808	276.1740	0.5168	0.5168	0.5233
81.7	279.0087	278.9741	283.3218	0.5312	0.5312	0.5379
81.8	285.6401	285.6137	290.0400	0.5450	0.5449	0.5518
81.9	291.8544	291.8357	296.3534	0.5581	0.5580	0.5650
82	297.6836	297.6718	302.2846	0.5706	0.5706	0.5778
83	339.5293	339.5445	344.5279	0.6721	0.6721	0.6799
84	360.2159	360.2310	364.5539	0.7436	0.7436	0.7504
85	366.9915	367.0008	370.3752	0.7956	0.7956	0.8009
86	364.1722	364.1757	366.6665	0.8343	0.8343	0.8383
87	354.5754	354.5750	356.3501	0.8636	0.8636	0.8665
88	340.1214	340.1189	341.3540	0.8861	0.8861	0.8881
89	322.1656	322.1625	323.0086	0.9034	0.9034	0.9048
90	301.6978	301.6948	302.2708	0.9169	0.9169	0.9179
91	279.4659	279.4633	279.8571	0.9274	0.9274	0.9280
92	256.0575	256.0556	256.3280	0.9355	0.9355	0.9358
93	231.9563	231.9550	232.1444	0.9417	0.9417	0.9419
94	207.5821	207.5811	207.7075	0.9463	0.9463	0.9464
95	183.3221	183.3214	183.3905	0.9496	0.9496	0.9496
96	159.5577	159.5573	159.5662	0.9518	0.9518	0.9517
97	136.6885	136.6884	136.6340	0.9530	0.9530	0.9527
98	115.1578	115.1582	115.0501	0.9532	0.9532	0.9529
99	95.4947	95.4958	95.3578	0.9525	0.9525	0.9522
100	78.3604	78.3614	78.1900	0.9510	0.9510	0.9506
101	64.2864	64.2879	64.0702	0.9486	0.9486	0.9481
102	53.1318	53.1320	52.8802	0.9452	0.9452	0.9448
103	44.2754	44.2747	44.0286	0.9409	0.9409	0.9404
104	37.1600	37.1594	36.9574	0.9355	0.9355	0.9351
105	31.3888	31.3885	31.2445	0.9289	0.9289	0.9285
106	26.6680	26.6681	26.5798	0.9209	0.9209	0.9206
107	22.7759	22.7763	22.7348	0.9112	0.9112	0.9112
108	19.5431	19.5436	19.5381	0.8997	0.8997	0.8999
109	16.8386	16.8391	16.8595	0.8860	0.8860	0.8866
110	14.5603	14.5608	14.5985	0.8696	0.8696	0.8708
111	12.6275	12.6279	12.6764	0.8499	0.8499	0.8519
112	10.9756	10.9760	11.0305	0.8263	0.8263	0.8293
113	9.5522	9.5524	9.6097	0.7978	0.7978	0.8020
114	8.3133	8.3135	8.3718	0.7632	0.7632	0.7688
115	7.2214	7.2214	7.2798	0.7209	0.7209	0.7280
116	6.2416	6.2417	6.2990	0.6685	0.6685	0.6771
117	5.3391	5.3391	5.3919	0.6030	0.6030	0.6121
118	4.4700	4.4700	4.5056	0.5182	0.5182	0.5264
118.1	4.3820	4.3820	4.4149	0.5081	0.5081	0.5162
118.2	4.2932	4.2932	4.3231	0.4977	0.4977	0.5057
118.3	4.2032	4.2032	4.2303	0.4868	0.4868	0.4948
118.4	4.1116	4.1116	4.1360	0.4754	0.4753	0.4835
118.5	4.0179	4.0179	4.0401	0.4634	0.4634	0.4718
118.6	3.9214	3.9214	3.9423	0.4507	0.4507	0.4595
118.7	3.8210	3.8210	3.8420	0.4373	0.4372	0.4468
118.8	3.7156	3.7156	3.7390	0.4228	0.4228	0.4335
118.9	3.6034	3.6034	3.6327	0.4073	0.4073	0.4195
119	3.4820	3.4819	3.5222	0.3904	0.3904	0.4049
119.1	3.3482	3.3481	3.4068	0.3718	0.3718	0.3894
119.2	3.1977	3.1977	3.2852	0.3511	0.3512	0.3730
119.3	3.0247	3.0247	3.1558	0.3279	0.3279	0.3555
119.4	2.8212	2.8212	3.0162	0.3013	0.3013	0.3366
119.5	2.5766	2.5766	2.8631	0.2706	0.2706	0.3161
119.6	2.2761	2.2761	2.6909	0.2345	0.2345	0.2932
119.7	1.9001	1.9001	2.4890	0.1917	0.1917	0.2670
119.8	1.4218	1.4218	2.2369	0.1402	0.1402	0.2355
119.9	0.8048	0.8048	1.9106	0.0774	0.0774	0.1957

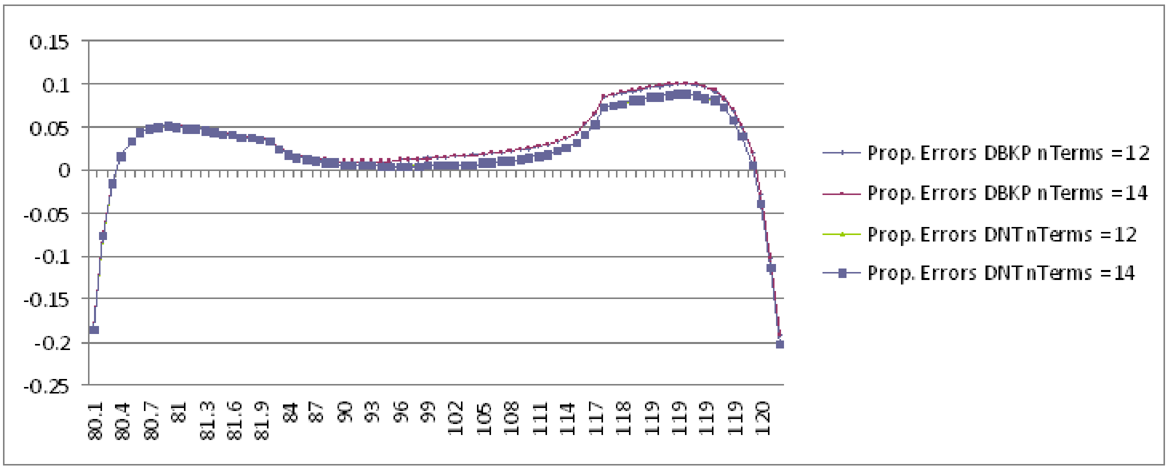
Table 22: CGMY double barrier option prices with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and $N = 14$.



(a) Double barrier knockout put option prices



(b) DNT option prices



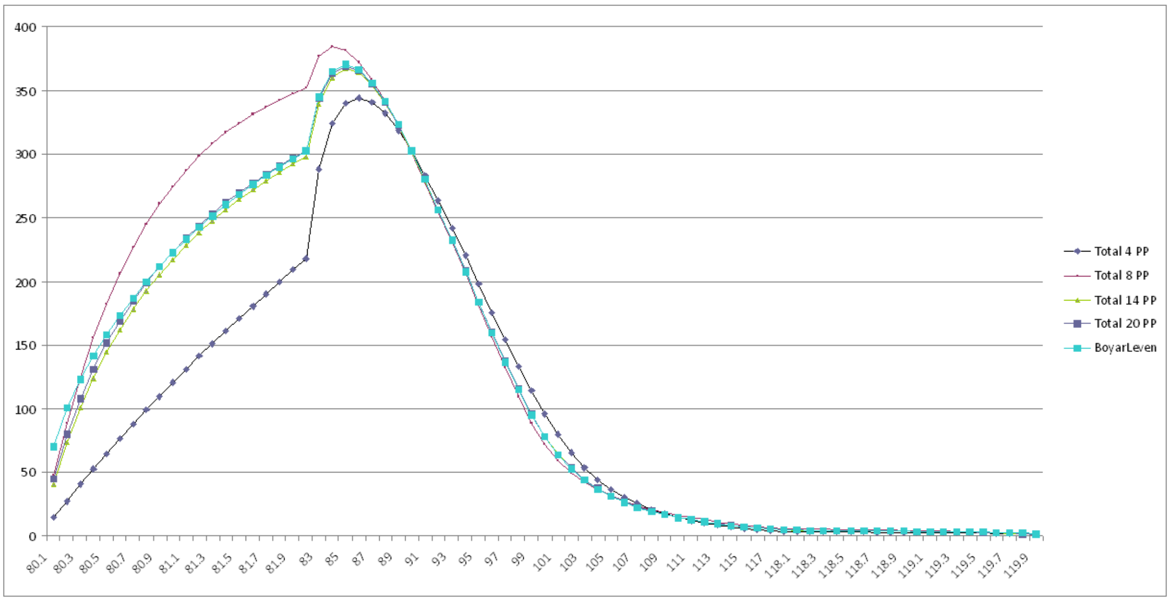
(c) Proportional errors for both double barrier knockout put (DBKP) option prices and double-no-touch (DNT) option prices

Figure 14: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ and $N = 14$.

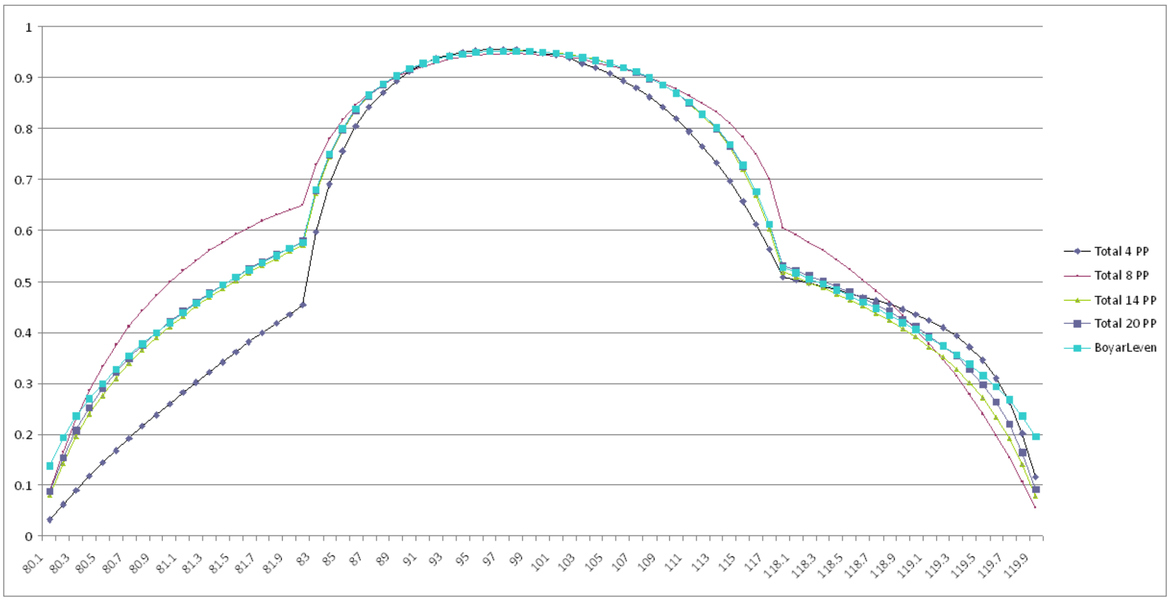
Spot as percentage of 3500	Double barrier knockout put options			DNT options		
	HEJDCC		BoyarLeven	HEJDCC		BoyarLeven
	nTerms = 12	nTerms = 14		nTerms = 12	nTerms = 14	
80.1	1.6344	1.6348	2.0117	0.0085	0.0085	0.0104
80.2	3.0487	3.0495	3.2965	0.0158	0.0158	0.0171
80.3	4.2980	4.2991	4.3685	0.0222	0.0222	0.0226
80.4	5.4226	5.4239	5.3373	0.0280	0.0280	0.0276
80.5	6.4521	6.4537	6.2409	0.0334	0.0334	0.0323
80.6	7.4084	7.4103	7.0989	0.0383	0.0383	0.0367
80.7	8.3078	8.3100	7.9227	0.0429	0.0430	0.0410
80.8	9.1624	9.1648	8.7195	0.0474	0.0474	0.0451
80.9	9.9812	9.9838	9.4943	0.0516	0.0516	0.0491
81	10.7711	10.7739	10.2509	0.0557	0.0557	0.0530
81.1	11.5373	11.5403	10.9918	0.0596	0.0596	0.0569
81.2	12.2838	12.2869	11.7192	0.0635	0.0635	0.0606
81.3	13.0137	13.0169	12.4347	0.0673	0.0673	0.0643
81.4	13.7293	13.7326	13.1395	0.0709	0.0710	0.0680
81.5	14.4325	14.4359	13.8348	0.0746	0.0746	0.0716
81.6	15.1249	15.1284	14.5214	0.0782	0.0782	0.0751
81.7	15.8075	15.8111	15.2000	0.0817	0.0817	0.0787
81.8	16.4815	16.4851	15.8712	0.0852	0.0852	0.0821
81.9	17.1476	17.1512	16.5355	0.0886	0.0886	0.0856
82	17.8063	17.8099	17.1934	0.0920	0.0920	0.0890
83	24.0776	24.0811	23.4779	0.1246	0.1246	0.1217
84	29.8990	29.9035	29.3233	0.1551	0.1551	0.1524
85	35.3324	35.3400	34.7784	0.1839	0.1839	0.1813
86	40.3849	40.3973	39.8470	0.2110	0.2110	0.2086
87	45.0473	45.0650	44.5187	0.2365	0.2365	0.2342
88	49.3064	49.3287	48.7793	0.2602	0.2602	0.2581
89	53.1493	53.1745	52.6157	0.2821	0.2821	0.2800
90	56.5659	56.5917	56.0175	0.3021	0.3021	0.3001
91	59.5492	59.5733	58.9782	0.3201	0.3202	0.3182
92	62.0961	62.1164	61.4954	0.3362	0.3362	0.3343
93	64.2070	64.2217	63.5705	0.3501	0.3502	0.3483
94	65.8855	65.8939	65.2087	0.3620	0.3620	0.3602
95	67.1390	67.1405	66.4186	0.3718	0.3718	0.3699
96	67.9774	67.9723	67.2117	0.3794	0.3794	0.3776
97	68.4133	68.4023	67.6023	0.3850	0.3850	0.3830
98	68.4615	68.4458	67.6064	0.3884	0.3884	0.3864
99	68.1387	68.1194	67.2418	0.3898	0.3897	0.3877
100	67.4628	67.4415	66.5274	0.3891	0.3891	0.3870
101	66.4530	66.4309	65.4831	0.3864	0.3864	0.3842
102	65.1288	65.1075	64.1290	0.3818	0.3817	0.3794
103	63.5106	63.4911	62.4853	0.3752	0.3752	0.3728
104	61.6182	61.6014	60.5720	0.3669	0.3669	0.3643
105	59.4715	59.4581	58.4087	0.3568	0.3568	0.3541
106	57.0898	57.0801	56.0139	0.3450	0.3450	0.3421
107	54.4912	54.4856	53.4055	0.3316	0.3316	0.3285
108	51.6931	51.6915	50.6000	0.3166	0.3166	0.3134
109	48.7115	48.7137	47.6123	0.3002	0.3002	0.2967
110	45.5606	45.5662	44.4557	0.2824	0.2825	0.2787
111	42.2527	42.2613	41.1415	0.2633	0.2634	0.2593
112	38.7975	38.8084	37.6783	0.2430	0.2430	0.2386
113	35.2010	35.2136	34.0713	0.2214	0.2214	0.2167
114	31.4644	31.4778	30.3210	0.1987	0.1987	0.1936
115	27.5805	27.5940	26.4208	0.1747	0.1747	0.1692
116	23.5280	23.5408	22.3520	0.1494	0.1494	0.1435
117	19.2562	19.2674	18.0726	0.1225	0.1225	0.1163
118	14.6296	14.6385	13.4859	0.0932	0.0932	0.0868
118.1	14.1352	14.1438	13.0030	0.0900	0.0901	0.0837
118.2	13.6325	13.6409	12.5143	0.0868	0.0869	0.0806
118.3	13.1206	13.1286	12.0193	0.0836	0.0836	0.0774
118.4	12.5982	12.6059	11.5176	0.0803	0.0803	0.0742
118.5	12.0638	12.0712	11.0084	0.0769	0.0769	0.0709
118.6	11.5160	11.5231	10.4909	0.0734	0.0734	0.0676
118.7	10.9526	10.9593	9.9643	0.0698	0.0698	0.0642
118.8	10.3712	10.3776	9.4274	0.0661	0.0661	0.0607
118.9	9.7690	9.7750	8.8789	0.0622	0.0622	0.0572
119	9.1425	9.1482	8.3171	0.0582	0.0583	0.0536
119.1	8.4877	8.4929	7.7399	0.0541	0.0541	0.0499
119.2	7.7993	7.8041	7.1447	0.0497	0.0497	0.0460
119.3	7.0715	7.0758	6.5276	0.0450	0.0450	0.0420
119.4	6.2967	6.3005	5.8838	0.0401	0.0401	0.0379
119.5	5.4660	5.4694	5.2061	0.0348	0.0348	0.0335
119.6	4.5687	4.5714	4.4831	0.0291	0.0291	0.0289
119.7	3.5913	3.5935	3.6948	0.0229	0.0229	0.0238
119.8	2.5179	2.5195	2.8051	0.0160	0.0160	0.0181
119.9	1.3288	1.3296	1.6456	0.0085	0.0085	0.0106

Table 23: CGMY double barrier option prices with $C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ and $N = 14$.

- Double barrier option price comparisons under CGMY with different N



(a) Double barrier knockout put option prices



(b) DNT option prices

Figure 15: CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and with different numbers of Poisson processes N.

Spot as percentage of 3500	Double barrier knockout put options					DNT options				
	HEJDCC				BoyarLeven	HEJDCC				BoyarLeven
	N = 4	N = 8	N = 14	N = 20		N = 4	N = 8	N = 14	N = 20	
80.1	14.0653	47.8103	40.3367	44.6168	70.5490	0.0317	0.0892	0.0790	0.0873	0.1370
80.2	27.5190	88.9146	73.2870	79.5693	100.8295	0.0618	0.1647	0.1426	0.1544	0.1942
80.3	40.4352	124.4666	100.7504	107.7998	122.9329	0.0903	0.2293	0.1948	0.2078	0.2353
80.4	52.8779	155.3693	124.0876	131.2746	141.5406	0.1174	0.2848	0.2387	0.2516	0.2697
80.5	64.9009	182.3380	144.2777	151.3180	158.0171	0.1434	0.3329	0.2764	0.2886	0.3000
80.6	76.5517	205.9478	162.0293	168.8251	172.9799	0.1683	0.3748	0.3092	0.3207	0.3275
80.7	87.8748	226.6674	177.8559	184.3988	186.7462	0.1923	0.4116	0.3385	0.3493	0.3529
80.8	98.9136	244.8834	192.1287	198.4442	199.5129	0.2154	0.4440	0.3648	0.3750	0.3765
80.9	109.7103	260.9194	205.1166	211.2359	211.4078	0.2378	0.4727	0.3889	0.3985	0.3986
81	120.3039	275.0496	217.0158	222.9655	222.5256	0.2596	0.4982	0.4110	0.4203	0.4194
81.1	130.7277	287.5104	227.9729	233.7736	232.9376	0.2809	0.5210	0.4316	0.4405	0.4391
81.2	141.0068	298.5079	238.1009	243.7689	242.7025	0.3017	0.5415	0.4507	0.4594	0.4577
81.3	151.1561	308.2239	247.4901	253.0398	251.8688	0.3220	0.5599	0.4687	0.4771	0.4753
81.4	161.1800	316.8192	256.2158	261.6606	260.4789	0.3420	0.5766	0.4857	0.4938	0.4921
81.5	171.0724	324.4363	264.3417	269.6945	268.5697	0.3616	0.5918	0.5017	0.5097	0.5081
81.6	180.8181	331.2009	271.9233	277.1956	276.1740	0.3808	0.6057	0.5168	0.5247	0.5233
81.7	190.3946	337.2230	279.0087	284.2104	283.3218	0.3996	0.6185	0.5312	0.5390	0.5379
81.8	199.7740	342.5984	285.6401	290.7788	290.0400	0.4180	0.6303	0.5450	0.5526	0.5518
81.9	208.9261	347.4096	291.8544	296.9347	296.3534	0.4359	0.6413	0.5581	0.5656	0.5650
82	217.8196	351.7271	297.6836	302.7077	302.2846	0.4534	0.6516	0.5706	0.5781	0.5778
83	287.9378	377.2446	339.5293	343.8006	344.5279	0.5975	0.7288	0.6721	0.6784	0.6799
84	324.3578	384.5436	360.2159	363.4334	364.5539	0.6913	0.7803	0.7436	0.7482	0.7504
85	340.0487	381.7261	366.9915	369.2497	370.3752	0.7555	0.8179	0.7956	0.7987	0.8009
86	344.2824	372.3766	364.1722	365.7072	366.6665	0.8037	0.8466	0.8343	0.8363	0.8383
87	340.9635	358.5261	354.5754	355.6201	356.3501	0.8414	0.8690	0.8636	0.8648	0.8665
88	332.0510	341.4390	340.1214	340.8634	341.3540	0.8710	0.8870	0.8861	0.8867	0.8881
89	318.8897	321.9652	322.1656	322.7432	323.0086	0.8943	0.9014	0.9034	0.9037	0.9048
90	302.5293	300.7141	301.6978	302.2060	302.2708	0.9124	0.9130	0.9169	0.9170	0.9179
91	283.8179	278.1482	279.4659	279.9654	279.8571	0.9265	0.9224	0.9274	0.9273	0.9280
92	263.4520	254.6357	256.0575	256.5821	256.3280	0.9372	0.9299	0.9355	0.9354	0.9358
93	242.0114	230.4841	231.9563	232.5179	232.1444	0.9451	0.9358	0.9417	0.9415	0.9419
94	219.9861	205.9636	207.5821	208.1745	207.7075	0.9507	0.9403	0.9463	0.9461	0.9464
95	197.7967	181.3280	183.3221	183.9231	183.3905	0.9543	0.9436	0.9496	0.9494	0.9496
96	175.8096	156.8388	159.5577	160.1314	159.5662	0.9561	0.9458	0.9518	0.9515	0.9517
97	154.3500	132.8086	136.6885	137.1891	136.6340	0.9564	0.9469	0.9530	0.9526	0.9527
98	133.7105	109.7512	115.1578	115.5370	115.0501	0.9553	0.9471	0.9532	0.9528	0.9529
99	114.1606	88.7864	95.4947	95.7084	95.3578	0.9528	0.9465	0.9525	0.9521	0.9522
100	95.9673	71.5984	78.3604	78.3977	78.1900	0.9490	0.9449	0.9510	0.9505	0.9506
101	79.4660	58.8652	64.2864	64.2398	64.0702	0.9438	0.9425	0.9486	0.9480	0.9481
102	65.1327	49.5421	53.1318	53.0999	52.8802	0.9372	0.9393	0.9452	0.9445	0.9448
103	53.3833	42.3189	44.2754	44.2875	44.0286	0.9291	0.9352	0.9409	0.9401	0.9404
104	44.0471	36.4624	37.1600	37.2207	36.9574	0.9195	0.9303	0.9355	0.9346	0.9351
105	36.5586	31.6005	31.3888	31.4893	31.2445	0.9081	0.9244	0.9289	0.9279	0.9285
106	30.4345	27.5046	26.6680	26.7948	26.5798	0.8949	0.9175	0.9209	0.9199	0.9206
107	25.3662	24.0175	22.7759	22.9157	22.7348	0.8797	0.9095	0.9112	0.9103	0.9112
108	21.1511	21.0251	19.5431	19.6851	19.5381	0.8622	0.9004	0.8997	0.8989	0.8999
109	17.6388	18.4413	16.8386	16.9751	16.8595	0.8423	0.8900	0.8860	0.8854	0.8866
110	14.7090	16.1994	14.5603	14.6862	14.5985	0.8197	0.8782	0.8696	0.8694	0.8708
111	12.2627	14.2460	12.6275	12.7401	12.6764	0.7942	0.8648	0.8499	0.8504	0.8519
112	10.2178	12.5373	10.9756	11.0742	11.0305	0.7654	0.8493	0.8263	0.8277	0.8293
113	8.5060	11.0369	9.5522	9.6374	9.6097	0.7332	0.8314	0.7978	0.8004	0.8020
114	7.0701	9.7136	8.3133	8.3870	8.3718	0.6972	0.8101	0.7632	0.7674	0.7688
115	5.8628	8.5391	7.2214	7.2861	7.2798	0.6571	0.7841	0.7209	0.7270	0.7280
116	4.8442	7.4865	6.2416	6.3004	6.2990	0.6126	0.7504	0.6685	0.6769	0.6771
117	3.9809	6.5206	5.3391	5.3941	5.3919	0.5632	0.7002	0.6030	0.6136	0.6121
118	3.2443	5.5430	4.4700	4.5222	4.5056	0.5082	0.6053	0.5182	0.5310	0.5264
118.1	3.1764	5.4338	4.3820	4.4343	4.4149	0.5022	0.5914	0.5081	0.5212	0.5162
118.2	3.1093	5.3201	4.2932	4.3459	4.3231	0.4962	0.5763	0.4977	0.5111	0.5057
118.3	3.0429	5.2010	4.2032	4.2566	4.2303	0.4899	0.5600	0.4868	0.5006	0.4948
118.4	2.9771	5.0753	4.1116	4.1663	4.1360	0.4835	0.5425	0.4754	0.4897	0.4835
118.5	2.9117	4.9418	4.0179	4.0747	4.0401	0.4768	0.5237	0.4634	0.4783	0.4718
118.6	2.8463	4.7988	3.9214	3.9811	3.9423	0.4698	0.5034	0.4507	0.4663	0.4595
118.7	2.7804	4.6446	3.8210	3.8850	3.8420	0.4623	0.4816	0.4373	0.4537	0.4468
118.8	2.7135	4.4771	3.7156	3.7855	3.7390	0.4542	0.4582	0.4228	0.4402	0.4335
118.9	2.6444	4.2940	3.6034	3.6813	3.6327	0.4453	0.4331	0.4073	0.4258	0.4195
119	2.5718	4.0923	3.4820	3.5703	3.5222	0.4352	0.4061	0.3904	0.4102	0.4049
119.1	2.4936	3.8689	3.3482	3.4499	3.4068	0.4236	0.3772	0.3718	0.3931	0.3894
119.2	2.4066	3.6198	3.1977	3.3161	3.2852	0.4097	0.3462	0.3511	0.3740	0.3730
119.3	2.3062	3.3408	3.0247	3.1629	3.1558	0.3929	0.3130	0.3279	0.3525	0.3555
119.4	2.1857	3.0267	2.8212	2.9818	3.0162	0.3719	0.2773	0.3013	0.3275	0.3366
119.5	2.0350	2.6717	2.5766	2.7602	2.8631	0.3450	0.2390	0.2706	0.2981	0.3161
119.6	1.8396	2.2687	2.2761	2.4797	2.6909	0.3099	0.1980	0.2345	0.2626	0.2932
119.7	1.5779	1.8100	1.9001	2.1131	2.4890	0.2634	0.1538	0.1917	0.2188	0.2670
119.8	1.2185	1.2863	1.4218	1.6208	2.2369	0.2009	0.1063	0.1402	0.1637	0.2355
119.9	0.7151	0.6871	0.8048	0.9448	1.9106	0.1161	0.0552	0.0774	0.0927	0.1957

Table 24: CGMY double barrier option prices with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and with different numbers of Poisson processes N.

- DNT price comparisons under CGMY with different N

Maturity	Lower barrier	Upper barrier	Tenor	Market prices	HEJDCC				BoyarLeven
					N = 4	N = 8	N = 14	N = 20	
0.08493	1.95	2.05	1m	0.765	0.885362	0.886110	0.875468	0.877142	0.877600
0.16986	1.95	2.05	2m	0.5	0.750916	0.749471	0.726646	0.726888	0.715400
0.25753	1.95	2.05	3m	0.325	0.612232	0.611078	0.581394	0.580084	0.558700
0.33699	1.95	2.05	4m	0.22	0.500713	0.500612	0.468705	0.466573	0.441100
0.41918	1.95	2.05	5m	0.15	0.403459	0.404403	0.372647	0.370137	0.343800
0.50685	1.95	2.05	6m	0.1	0.318989	0.320784	0.290759	0.288164	0.263000
0.75616	1.95	2.05	9m	0.05	0.161805	0.164501	0.142474	0.140377	0.122100
1.00548	1.95	2.05	12m	0.03	0.081557	0.083935	0.069518	0.068107	0.056500
0.08493	1.97	2.04	1m	0.515	0.821259	0.810722	0.787550	0.786294	0.774600
0.25753	1.97	2.04	3m	0.115	0.413334	0.411533	0.381523	0.376341	0.338100
0.01918	1.98	2.03	1w	0.85	0.953346	0.944942	0.934004	0.933326	0.933400
0.08493	1.98	2.03	1m	0.245	0.695626	0.692064	0.664399	0.656565	0.619300

Table 25: CGMY DNT option prices for $C = 0.00954,\ G = 9.99214,\ M = 10.75716,\ Y = 1.25$ and with different numbers of Poisson processes N.

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