VARIANCE DERIVATIVES: PRICING AND CONVERGENCE

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Abstract. We examine the pricing of variance swaps and some generalisations and variants such as self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps. We consider the pricing of both discretely monitored and continuously monitored versions of these swaps when the dynamics of the log of the underlying stock price are driven by (possibly, multiple) time-changed Lévy processes. Using the method of Hong (2004), we derive exact and easily implementable formulae for the prices of these swaps in terms of, essentially, the characteristic function of the log of the stock price and its derivatives. We consider the convergence of the prices of discretely monitored versions of these swaps to the prices of their continuously monitored counterparts as the number N of monitoring times is allowed to tend to infinity. We significantly extend results in Broadie and Jain (2008a) by showing that the prices of discretely monitored variance swaps and all of the generalisations and variants listed above all converge to the prices of their continuously monitored counterparts as $O(1/N)$. We generalise results in Carr and Lee (2009) by relating the prices of variance swaps and the generalisations and variants listed above to the prices of log-forward-contracts and entropy-forward-contracts. Carr and Lee (2009) show that, under an independence assumption, discretely monitored variance swaps are worth at least as much as their continuously monitored counterparts. We extend this result in two directions, by dropping the independence assumption and by proving analogous results for some of the other swaps listed above.

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1. Introduction

The variance of the price returns of a stock or of a stock index plays a crucial role in financial economics because of its fundamental role in determining the risk of a stock portfolio (Markowitz (1952)), the equilibrium excess return on a stock (Sharpe (1964)) and in option pricing (Black and Scholes (1973), Merton (1973)). Therefore, it is not surprising that, in recent years, variance swaps (and other derivative contracts whose payoffs are linked to measures of realised variance) have attracted considerable interest from academics and practitioners alike.

This paper examines the pricing of variance derivatives. In particular, we consider the pricing of discretely monitored variance swaps - that is to say, variance swaps which have, for example, daily or weekly monitoring. We examine the convergence of the prices of these swaps to the prices of their continuously monitored counterparts. In fact, we go beyond this and focus our attention upon the pricing of what we call "generalised variance swaps", which include standard variance swaps, self-quantoed variance swaps and gamma swaps as special cases. We also consider skewness swaps and proportional variance swaps.

Nearly all papers on variance swaps have focussed on the log-contract replication approach (Neuberger (1990), (1994), (1996), Britten-Jones and Neuberger (2000), Dupire (1993), Demeterfi et al. (1999), Broadie and Jain (2008b), Carr and Lee (2010a)). In essence, this approach works by noting that, under the assumption that the stock price process has continuous sample paths, the payoff of a variance swap can be replicated by a static position of being short two log-contracts and by a dynamic position of being long $2/S(t)$ units of stock, where $S(t)$ is the stock price at time t. In the assumed absence of arbitrage, this strategy also yields the price of a variance swap. This approach actually values continuously monitored variance swaps. In practice, variance swaps are discretely monitored - usually there is daily monitoring but contracts with weekly or monthly monitoring trade as well. As well as valuing continuously monitored rather than discretely monitored variance swaps, the log-contract replication approach has an additional disadvantage - it assumes that the stock price has continuous sample paths. But almost every major empirical study (see Broadie et al. (2007), Carr et al. (2002) and the references therein) has indicated the necessity of incorporating jumps into the dynamics of stock prices. In recognition of the empirical evidence, we assume, throughout the paper, that the log of the stock price is driven by (possibly, multiple) time-changed Lévy processes. The time-change processes can be deterministic or stochastic. The latter allows us to model, for example, volatility clustering and leverage effects which are well-known (Carr et al. (2003)) to be observed in stock price returns. The Lévy processes can essentially be arbitrary (subject to minor regularity conditions).

There is a completely different methodology for valuing variance swaps which, compared to the log-contract replication approach, has two major benefits: It values discretely monitored variance swaps and it works even when there are jumps in the dynamics of the underlying stock price. The only reference we knew to this methodology at the time of starting to write the first draft of this paper is an unpublished seminar presentation, Hong (2004), given at Cambridge University. The methodology we will present in this paper builds upon the approach of Hong (2004). To give the reader an early taste of our paper, we now briefly describe how variance swaps can be priced using the Hong (2004) methodology.

We denote the initial time (today) by $t_0 \equiv 0$. We denote the stock price, at time t, by $S(t)$. The variance swap is written at time t_0 and matures at time T. The time interval $[t_0, T]$ is partitioned into N time periods whose end-points are t_j, $j = 1, 2, ..., N$, where $0 \equiv t_0 < t_1 < ... < t_{j-1} < t_j < ... < t_N \equiv T$. We will **not** assume that the time periods $t_j - t_{j-1}$ are equal in length although, in practice, they are often approximately equal.

We assume that the stock price $S(t)$, at time t, under some equivalent martingale measure $\mathbb{Q}$, follows:

$$S(t) = S(t_0)\exp(\int_{t_0}^{t}(r(s)-q(s))ds + L_t), \qquad L_{t_0} \equiv 0, \text{ with } \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(L_t)] = 1, \text{ for all } t \geq t_0,$$

where L_t is some process, such as, for example, a Lévy process. Here $r(t)$ and $q(t)$ are the risk-free interest-rate and the dividend yield, at time t, which in this section only, we allow to be stochastic. Later on, we will restrict the risk-free interest-rate and the dividend yield to be deterministic for simplicity.

Define the extended discounted characteristic function $\Phi(z; t_j, t_{j-1})$ as follows:

$$\Phi(z; t_j, t_{j-1}) \quad \equiv \quad \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-\int_{t_0}^{T} r(s)ds)\exp(iz\log(S(t_j)/S(t_{j-1})))],$$

where z may be real or complex (provided the expectation is well-defined). For models where the interest-rates and dividend yields follow, for example, processes of the Hull-White, gaussian HJM or affine type (see Duffie et al. (2000)) and L_t is a Lévy or affine jump-diffusion process, then $\Phi(z; t_j, t_{j-1})$ will usually be known analytically. Then note:

$$-\frac{\partial^2\Phi(z; t_j, t_{j-1})}{\partial z^2} \quad = \quad \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-\int_{t_0}^{T} r(s)ds)(\log(S(t_j)/S(t_{j-1})))^2\exp(iz\log(S(t_j)/S(t_{j-1})))].$$

Evaluating the last equation at $z = 0$ and summing from $j = 1$ to N, we get:

$$\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-\int_{t_0}^{T} r(s)ds)\sum_{j=1}^{N}(\log(S(t_j)/S(t_{j-1})))^2] = -\sum_{j=1}^{N}\frac{\partial^2\Phi(z; t_j, t_{j-1})}{\partial z^2}|_{z=0}.$$

The left-hand-side of this equation is clearly the price, at time t_0, of the floating leg of a discretely monitored variance swap. But we know $\Phi(z; t_j, t_{j-1})$ and hence the right-hand-side for many models of interest.

This pricing methodology is generic and can be used for almost all stochastic processes that have been used in mathematical finance (with the exception of local volatility models because, in this case, $\Phi(z; t_j, t_{j-1})$ is not known) with the process parameters typically being obtained by a calibration to the market prices of vanilla options.

Recently, another paper, Itkin and Carr (2010), has appeared which independently builds upon the Hong (2004) methodology. However, our paper is significantly different from Itkin and Carr (2010). They focus upon deriving approximate prices for variance swaps when the time interval between successive monitoring dates is very small (but not infinitesimal) or, equivalently, deriving approximate prices when the number of monitoring times is very large (but not infinite). By contrast, we do not use any approximations. Compared to Itkin and Carr (2010), we also consider a potentially wider class of stochastic processes as well as additional types of swaps.

Our work complements and significantly extends that of Broadie and Jain (2008a) and that of Carr and Lee (2009), although each paper uses very different methodologies.

- We extend results in Broadie and Jain (2008a) connected with pricing discretely monitored variance swaps and the the convergence of these prices to their continuously monitored counterparts in three important ways.
  - We consider generalised variance swaps which include standard variance swaps, self-quantoed variance swaps and gamma swaps as special cases.
  - We also consider skewness swaps and proportional variance swaps (defined in Section 2).
  - We work with a much wider class of underlying processes.

- Carr and Lee (2009), working in continuous time, show that the price of a continuously monitored variance swap is equal to minus Q_X^{VS} times the price of a log-forward-contract, where Q_X^{VS} is termed the "multiplier", and is equal to two if the (log of the) underlying stock price process is a (possibly, time-changed) Brownian motion but is some number (in general) different from two if the (log of the) stock price process is discontinuous - for example, a (possibly, time-changed) Lévy process.
  - We derive the same result by a completely different methodology but, in addition, we extend the result by allowing for multiple Lévy processes with multiple time-change processes.
  - We also extend the result to other types of variance swaps.

These results have assumed greater significance since the global financial crisis of Autumn 2008. Before the financial crisis, market prices of variance swaps were close to minus two times the prices of log-forward-contracts (where the latter were inferred from the market prices of co-terminal vanilla options of as many strikes as were available as described in, for example, Carr and Madan (1998) and Demeterfi et al. (1999)). This is the same as saying that variance swaps traded in line with the classical log-contract replication approach of Neuberger (1990) (we might conjecture that there was an element of a self-fulfilling prophecy in this) or, in terms of Carr and Lee (2009), it is saying that before the financial crisis, the "empirical" value (i.e. the value implied by dividing the market prices of variance swaps by (minus) the prices of log-forward-contracts inferred from the market prices of vanilla options) of the multiplier Q_X^{VS} was approximately two. In the aftermath of the financial crisis, this is no longer the case. Traders have reported to us that, since the financial crisis, for variance swaps written on most major stock indices, the empirical value (i.e. the value implied by market prices) of the multiplier Q_X^{VS} has been consistently in the range 2.10 to 2.25 (which is a significant departure from two, especially taking into account that variance swaps are usually considered to be simple "flow" derivatives rather than highly exotic derivatives). This is illustrated in figure 1, where we have displayed the empirical value of Q_X^{VS} based on market transactions recorded on 10th December 2010 for variance swaps written on the Nikkei-225 stock index with maturities between one month and six months. We thank James Battle of UBS for providing us with this data. We see in figure 1 that the empirical value of Q_X^{VS} for the Nikkei-225 is approximately 2.17. Carr and Lee (2009) show (see also Section 7) that this value of Q_X^{VS} is consistent with a significantly negatively skewed Lévy process (i.e. one with larger down jumps) which is, in turn, consistent with the negatively skewed implied volatility surfaces empirically observed for options on stocks and stock indices. Intuitively speaking, this may suggest that, since the financial crisis of 2008, markets price variance swaps taking into account the possibility of large downward jumps in the underlying stock price process.

Whilst we will focus upon the pricing of variance derivatives in this paper, we now briefly mention, in passing, two further possible applications of the results we will present here. Firstly, Carr and Wu (2009) show that stock markets have a negative variance risk premium by computing the difference between realised variance under the real world physical measure and the expected risk-neutral variance under an equivalent martingale measure $\mathbb{Q}$ (which is computed via its relationship with log-forward-contracts which in turn are synthesised from co-terminal vanilla options). The former is implicitly discretely monitored whilst the latter is implicitly continuously monitored. Secondly, there is currently considerable interest (see, for example, Ait-Sahalia (2004) and Barndorff-Nielsen et al. (2008) and the references therein) in estimating process parameters (under the real world physical measure) using high-frequency estimates of realised variance. We hope that our results may find use in both of these applications (steps in that direction can be found in Crosby (2010), which is a talk based on this paper) but they are both beyond the scope of the current paper.

The rest of this paper is structured as follows: In Section 2, we introduce some notation and define the payoffs of the swaps that we will consider in this paper. In Section 3, we specify the dynamics of the underlying stock. In Section 4, we price generalised variance swaps, while we consider the special case of variance swaps in Section 5. In Section 6, we extend by considering the limit as the number of monitoring dates increases to infinity. In Section 7, we consider the relationship between variance swaps and log-forward-contracts, and more generally the relationship between the generalisations and variants listed above and the prices of log-forward-contracts and entropy-forward-contracts. In Section 8, we illustrate with some numerical examples. In Sections 1 to 8, we assume that the time-change process(es) are always continuous. In Section 9, we briefly examine the consequences of relaxing that assumption by allowing the time-change processes to be discontinuous. Section 10 concludes.

2. Notation and contract definitions

In this section, we define contract payoffs and terminology.

It is clear that the methodology described in Section 1 for pricing variance swaps works even if the risk-free interest-rate and the dividend yield are allowed to be stochastic. However, in practice, variance swaps and related derivatives traded in the markets are typically not very long-dated, and we might expect the impact of stochastic risk-free interest-rates and stochastic dividend yields to be small. For this reason and for simplicity, we henceforth assume, throughout the rest of this paper, that the risk-free interest-rate $r(t)$ and the dividend yield $q(t)$, at time t, are deterministic. Hence, we can take discounting terms like $\exp(-\int_{t_0}^{T} r(s)ds)$ outside expectations. We define $P(t_0, t)$ to be the price of a zero-coupon bond, at time t_0, that matures at time t. Clearly, $P(t_0, t) \equiv \exp(-\int_{t_0}^{t} r(s)ds)$ for all $t \geq t_0$. For any (possibly complex) z and for any times $t_0 \leq t_{\text{start}} \leq t_{\text{end}}$, we define $D(t_{\text{start}}, t_{\text{end}}, iz)$ via:

$$D(t_{\text{start}}, t_{\text{end}}, iz) \equiv \exp(iz \int_{t_{\text{start}}}^{t_{\text{end}}} (r(s) - q(s))ds). \tag{2.1}$$

We assume that $r(t)$ and $q(t)$ are finite, for all $t \geq t_0$.

We denote the forward stock price, at time t, to time T, by $F(t, T)$, with $F(t, T)$ given by:

$$F(t, T) \equiv S(t) \exp(\int_{t}^{T} (r(s) - q(s))ds) = S(t)D(t, T, 1). \tag{2.2}$$

Recall that we work with a time partition: $0 \equiv t_0 < t_1 < ... < t_{j-1} < t_j < ... < t_N \equiv T$.

We define a (discretely monitored) "generalised variance swap" to have a (floating leg) payoff at time T equal to

$$\sum_{j=1}^{N} \left(\log \frac{S(t_j)}{S(t_{j-1})}\right)^2 \left(\frac{S(t_{j-1})}{S(t_0)}\right)^{\Upsilon_1} \left(\frac{S(t_j)}{S(t_{j-1})}\right)^{\Upsilon_2} \left(\frac{S(t_N)}{S(t_j)}\right)^{\Upsilon_3} \quad \text{and we let its price,} \tag{2.3}$$

at time t_0, be $\text{GVS}(t_0, T, N, \Upsilon_1, \Upsilon_2, \Upsilon_3)$. Note that Υ_1, Υ_2 and Υ_3 can be any real numbers for which the price $\text{GVS}(t_0, T, N, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ is finite and well-defined. Generalised variance swaps include variance swaps, self-quantoed variance swaps and gamma swaps as special cases.

For example, a variance swap pays $\sum_{j=1}^{N} \left(\log \frac{S(t_j)}{S(t_{j-1})}\right)^2$ at time T and its price, at time t_0, is $\text{VS}(t_0, T, N) \equiv \text{GVS}(t_0, T, N, 0, 0, 0)$. Similarly, a self-quantoed variance swap (respectively, a gamma swap) pays $\sum_{j=1}^{N} \left(\frac{S(T)}{S(t_0)}\right) \left(\log \frac{S(t_j)}{S(t_{j-1})}\right)^2$ (respectively, $\sum_{j=1}^{N} \left(\log \frac{S(t_j)}{S(t_{j-1})}\right)^2 \left(\frac{S(t_j)}{S(t_{j-1})}\right)$) at time T and its price, at time t_0, is $\text{GVS}(t_0, T, N, 1, 1, 1)$ (respectively, $\text{GVS}(t_0, T, N, 0, 1, 0)$). Note that the reader should check our definition because there are alternative definitions for gamma swaps. Furthermore, Overhaus et al. (2007) refer to our definition of a gamma swap as an "entropy swap".

We also define three more variance-related contracts - none of which is a special case of equation (2.3).

For $\varphi = 0, 1$, a φ-log-entropy-forward-contract has a payoff at time T

$$\left(\frac{F(T,T)}{F(t_0,T)}\right)^{\varphi} \log \frac{F(T,T)}{F(t_0,T)} \qquad \text{and its price, at time } t_0, \text{ is } \mathrm{LEFC}(t_0,T,\varphi). \tag{2.4}$$

When $\varphi = 0$ (respectively, $\varphi = 1$), a φ-log-entropy-forward-contract coincides with a log-forward-contract (respectively, an entropy-forward-contract).

A (discretely monitored) skewness swap has a payoff at time T

$$\sum_{j=1}^{N} \big(\log \frac{S(t_j)}{S(t_{j-1})}\big)^3 \qquad \text{and its price, at time } t_0, \text{ is } \mathrm{SKS}(t_0,T,N). \tag{2.5}$$

A (discretely monitored) proportional variance swap has a payoff at time T

$$\sum_{j=1}^{N} \big(\frac{S(t_j)}{S(t_{j-1})} - 1\big)^2 \qquad \text{and its price, at time } t_0, \text{ is } \mathrm{PVS}(t_0,T,N). \tag{2.6}$$

For all of the above swaps, we have defined the discretely monitored version. The continuously monitored version is, essentially (we defer detailed discussion until Section 6 because we do not need them until then), that obtained by letting $N \to \infty$ (with $\sup\{t_j - t_{j-1}\} \to 0$). We will denote the prices of the continuously monitored version of these swaps by the same notation but with N replaced by ∞. So, for example, $\mathrm{VS}(t_0,T,\infty) \equiv \mathrm{GVS}(t_0,T,\infty,0,0,0)$ is the price, at time t_0, of a continuously monitored variance swap. Note that for all the swaps defined above, we have defined the floating leg. There is typically a fixed leg (which we will omit in our analysis), determined as that which makes the swap have a net value of zero, at time t_0.

3. Stock price dynamics: Model set-up

In this section, we define the dynamics of the underlying stock.

We make the following assumption throughout this paper.

Assumption 3.1. The market is free of arbitrage. More specifically, we assume that there is "no free lunch with vanishing risk" (Delbaen and Schachermayer (1998)). This guarantees the existence of a risk-neutral equivalent martingale measure which will, in general, not be unique. If non-unique, we will assume a specific risk-neutral equivalent martingale measure, denoted by $\mathbb{Q}$, has been chosen (possibly, via a calibration to market prices).

We begin with a filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq t_0}, \mathbb{Q})$ satisfying the usual conditions. We assume that we have K independent non-constant Lévy processes, with respect to $\{\mathcal{G}_t\}$, denoted by $\tilde{X}_t^{(k)}$, for $k = 1, 2, \ldots, K$, satisfying $\tilde{X}_{t_0}^{(k)} = 0$, having characteristic exponent $\tilde{\psi}_k(u)$.

Assumption 3.2. For each $k = 1, 2, \ldots, K$, we define $X_t^{(k)} \equiv \tilde{X}_t^{(k)} + (t - t_0)\tilde{\psi}_k(-i)$. This implies that, for each $k = 1, 2, \ldots, K$, $\exp(X_t^{(k)})$ is a non-constant martingale, under $\mathbb{Q}$, with respect to the natural filtration generated by $X_t^{(k)}$ i.e. that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t^{(k)})] = \exp(X_{t_0}^{(k)}) = 1$ for all $t \geq t_0$. We assume that that exists a pair of numbers A and B with $A \geq 2$, $B \leq 0$ such that for all Υ satisfying $B \leq \Upsilon \leq A$, and for all $k = 1, 2, \ldots, K$,

$\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(\Upsilon X^{(k)}_t)]$, $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(\Upsilon X^{(k)}_t)X^{(k)}_t]$ $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(\Upsilon X^{(k)}_t)X^{(k)2}_t]$ and $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(\Upsilon X^{(k)}_t)X^{(k)3}_t]$ are finite, for some $t > t_0$. We write the characteristic function of $X^{(k)}_t$ as

$$\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iuX^{(k)}_t)] \quad \equiv \quad \exp(-(t-t_0)\psi_k(u)), \quad \text{where } \psi_k(u) \equiv \tilde{\psi}_k(u) - iu\tilde{\psi}_k(-i). \tag{3.1}$$

The assumption of independence means that $X^{(k)}_t$ is independent of $X^{(\ell)}_t$, if $k \neq \ell$.

For each k, $\psi_k(u)$ is the mean-corrected characteristic exponent.

For each $k = 1, 2, \ldots, K$, we let the Lévy processes $X^{(k)}_t$ have Brownian variance $\sigma^{(k)\,2}$, Lévy measure $\nu^{(k)}$ and purely continuous and purely discontinuous parts $X^{c,(k)}_t$ and $X^{d,(k)}_t$ respectively, i.e. $X^{(k)}_t$ is the sum of $X^{c,(k)}_t + X^{d,(k)}_t$ (plus a deterministic drift term). We write $\Delta X^{d,(k)}_t = X^{d,(k)}_t - X^{d,(k)}_{t-}$.

We collect together some essential results concerning $X^{(k)}_t$ and $\psi_k(u)$ in the following lemma:

Lemma 3.3. *For future reference, we define, for each k, the deterministic quantity*

$$m_k(iu) \equiv i\tilde{\psi}'_k(u) + \tilde{\psi}_k(-i) = i\psi'_k(u), \quad \textit{where } ' \textit{ denotes differentiation i.e.} \tag{3.2}$$

$\tilde{\psi}'_k(u) \equiv \partial\tilde{\psi}_k(u)/\partial u$. *For any real Υ, $\exp(\Upsilon X_t)X^{(k)}_t$ and $\psi_k(-i\Upsilon) = (\tilde{\psi}_k(-i\Upsilon) - \Upsilon\tilde{\psi}_k(-i))$ are real. Furthermore:*

$$m_k(\Upsilon) = i\tilde{\psi}'_k(-i\Upsilon) + \tilde{\psi}_k(-i) = i\psi'_k(-i\Upsilon) \quad \textit{is real, for any real } \Upsilon. \tag{3.3}$$

Further, $m_k(0)$ is strictly negative and $m_k(1)$ is strictly positive for any non-constant Lévy process $X^{(k)}_t$. For any (non-constant) Lévy process $X^{(k)}_t$:

$$\begin{aligned} &\tilde{\psi}_k(0) = 0, \ \psi_k(0) = 0, \ \psi_k(-i) = 0, \ \psi_k(-2i) < 0, \\ &\tilde{\psi}''_k(-i\Upsilon) = \psi''_k(-i\Upsilon) > 0, \quad \textit{for any real } \Upsilon, \\ &\mathbb{E}^{\mathbb{Q}}_{t_0}[X^{(k)}_t - X^{(k)}_{t_0} - (t-t_0)m_k(0)] = 0. \end{aligned} \tag{3.4}$$

Proof: The first part is obvious. By differentiating the characteristic function of $X^{(k)}_t$ with respect to u and setting $iu = \Upsilon$, where Υ is real, we get

$$\begin{aligned} i\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(\Upsilon X^{(k)}_t)X^{(k)}_t] &= -(t-t_0)(\tilde{\psi}'_k(-i\Upsilon) - i\tilde{\psi}_k(-i))\exp(-(t-t_0)\psi_k(-i\Upsilon)) \\ &= i(t-t_0)m_k(\Upsilon)\exp(-(t-t_0)\psi_k(-i\Upsilon)). \end{aligned}$$

The second part follows. The concavity of the log function together with the fact that $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(X^{(k)}_t)] = 1$ allows us to conclude that $\mathbb{E}^{\mathbb{Q}}_{t_0}[X^{(k)}_t] < 0$. Hence $m_k(0) < 0$. By applying Jensen's inequality to the convex function $f(x) = x\log x$ with $x = \exp(X^{(k)}_t)$, we see that $m_k(1) > 0$. The rest is standard. •

We will time-change each of the K Lévy processes $X^{(k)}_t$ by a time-change process $Y^{(k)}_t$, $k = 1, 2, \ldots, K$.

Assumption 3.4. We assume that we have K (possibly, deterministic) increasing time-change processes denoted by $Y^{(k)}_t$, for each $k = 1, 2, \ldots, K$, each of which is a family of $\{\mathcal{G}_t\}$ stopping times.

We assume that, for each $k = 1, 2, \ldots, K$, $Y^{(k)}_{t_0} = t_0 \equiv 0$, $Y^{(k)}_t$ is finite $\mathbb{Q}-$ a.s., $\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T] < \infty$ and $\mathbb{V}ar^{\mathbb{Q}}_{t_0}[Y^{(k)}_T] < \infty$ (for $T < \infty$) and that $Y^{(k)}_t \to \infty$ as $t \to \infty$.

Now we construct the price process for the stock price. We have K Lévy processes each satisfying Assumption (3.2) with $X^{(k)}_{t_0} = 0$. We have K time-change processes, denoted by $Y^{(k)}_t$, for $k = 1, 2, \ldots, K$, each satisfying Assumption (3.4) with $Y^{(k)}_{t_0} = t_0$. For each $k = 1, 2, \ldots, K$, we time-change the Lévy process

$X_t^{(k)}$ by $Y_t^{(k)}$ to get a process $X_{Y_t^{(k)}}^{(k)}$ which we henceforth denote by $X_{Y_t}^{(k)}$. Note that for each $k = 1, 2, \ldots, K$, $X_{Y_{t_0}}^{(k)} = 0$.

We now define the dynamics of the stock price:

Assumption 3.5. For $t \geq t_0$, the stock price process $S(t)$, under the equivalent martingale measure $\mathbb{Q}$, is assumed to be:

$$S(t) = S(t_0) \exp\left(\int_{t_0}^{t} (r(s) - q(s)) ds + \sum_{k=1}^{K} X_{Y_t}^{(k)}\right). \tag{3.5}$$

We assume that the forward stock price $F(t, T)$ is a martingale, under $\mathbb{Q}$, with respect to the filtration generated by $\mathcal{F}_t \equiv \sigma\{X_{Y_v}^{(1)}, X_{Y_v}^{(2)}, \ldots, X_{Y_v}^{(K)}, v \leq t\}$. That is,

$$\mathbb{E}_t^{\mathbb{Q}}[F(u, T)] = F(t, T), \text{ for all } u \in [t, T], \tag{3.6}$$

where, in the last equation, the expectation is conditional on $\mathcal{F}_t$. From now on whenever we write an expectation in the form $\mathbb{E}_t^{\mathbb{Q}}[\bullet]$, we mean that the expectation is conditional on the filtration $\mathcal{F}_t$.

Remark 3.6. The stock price process assumed in equation (3.5) is very general and includes most models used in mathematical finance. In some models of practical interest, such as the VG-CIR, VG-OU, CGMY-CIR, CGMY-OU models described in Section 7.3 of Schoutens (2003) (and originally due to Carr et al. (2003)), K is equal to one. We have allowed K to be general in order to include the class of stochastic skew models of Carr and Wu (2007), the "stochastic volatility plus jumps" model of Bates (1996) as well as the non-Gaussian OU-based models of Barndorff-Nielsen and Shephard (2001). (Carr and Wu (2007) (see their equation (39)) explicitly reconstruct the Bates (1996) model in time-changed Lévy process form whilst Carr et al. (2003) do likewise for the Barndorff-Nielsen and Shephard (2001) model.)

We now define, for all $t \geq t_0$

$$\Xi_t(u) \equiv \exp\left(\sum_{k=1}^{K} \Big(iuX_{Y_t}^{(k)} + (Y_t^{(k)} \psi_k(u))\Big)\right). \tag{3.7}$$

We also introduce what we call the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$, which we define, for each j, $j = 1, \ldots, N$, by

$$\begin{aligned}
\Phi(z_1, z_2, z_3; j) &\equiv \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iz_1 \log \frac{S(t_{j-1})}{S(t_0)} + iz_2 \log \frac{S(t_j)}{S(t_{j-1})} + iz_3 \log \frac{S(t_N)}{S(t_j)})] \\
&= D(t_0, t_{j-1}, iz_1) D(t_{j-1}, t_j, iz_2) D(t_j, t_N, iz_3) \times \\
&\quad \mathbb{E}_{t_0}^{\mathbb{Q}}\Big[\exp(iz_1 \sum_{k=1}^{K} X_{Y_{t_{j-1}}}^{(k)}) \mathbb{E}_{t_{j-1}}^{\mathbb{Q}}[\frac{\Xi_{t_j}(z_2)}{\Xi_{t_{j-1}}(z_2)} \exp(-\sum_{k=1}^{K} (Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)}) \psi_k(z_2)) \\
&\quad \times \mathbb{E}_{t_j}^{\mathbb{Q}}[\exp(iz_3 \sum_{k=1}^{K} (X_{Y_{t_N}}^{(k)} - X_{Y_{t_j}}^{(k)}))]]\Big],
\end{aligned} \tag{3.8}$$

where $D(t_0, t_{j-1}, iz_1)$, $D(t_{j-1}, t_j, iz_2)$ and $D(t_j, t_N, iz_3)$ are defined via equation (2.1).

Assumption 3.7. We assume that $\Xi_t(u)$ is a martingale, under $\mathbb{Q}$, with respect to the filtration generated by $\mathcal{F}_t \equiv \sigma\{X_{Y_v}^{(1)}, X_{Y_v}^{(2)}, \ldots, X_{Y_v}^{(K)}, v \leq t\}$ when u is imaginary and of the form $u = -ic$ where $c \in \mathbb{R}$ lies in the range $c \in [-2, 0]$. We assume that the payoffs of all the contracts defined in Section 2 have finite expectation under $\mathbb{Q}$. Similarly, in equation (3.8), we assume that iz_1, iz_2 and iz_3 all lie in a region such that the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$ is finite and well-defined. Finally, we assume that $\Xi_t(u)$ and $\Phi(z_1, z_2, z_3; j)$ are sufficiently well-behaved that we can exchange the order of expectation and differentiation (see equations (3.10), (4.2) and (4.5)).

Of course, it would be nice to state a set of assumptions concerning $X_t^{(k)}$ and $Y_t^{(k)}$, for each $k = 1, 2, \ldots, K$, which imply that Assumption (3.7) is true. However, even in the case of a single Lévy process for which all exponential moments exist (eg. Brownian motion) time-changed by an integrated Heston (1993) process, we know (see Andersen and Piterbarg (2007)) that it is a non-trivial exercise to relate these. This is likely to be even more challenging for Lévy processes where some exponential moments are infinite and for general time-change processes. Since this is not the purpose of the present paper, we restrict ourselves by making Assumption (3.7). We note however that Carr and Wu (2004) prove that $\Xi_t(u)$ is a martingale, under $\mathbb{Q}$, in the special cases that u is real and that there is a common time change process (in the sense of Assumption (3.9) - see later in this section). Their argument is based on noting that, for each $k = 1, 2, \ldots, K$, (by re-arranging equation (3.1)):

$$\exp(iuX_t^{(k)} + (t - t_0)\psi_k(u)) \quad \text{is a martingale, under } \mathbb{Q} \ ,$$

and then replacing calendar time by a stochastic time-change process (which is justified by the optional stopping theorem). Our more general case will typically require, for example, parameter restrictions (which will vary on a case-by-case basis - as already alluded to, Andersen and Piterbarg (2007) detail the parameter restrictions for Brownian motion time-changed by an integrated Heston (1993) process and also discuss several other stochastic volatility models) to prove that $\Xi_t(u)$ is a martingale, under $\mathbb{Q}$.

By Assumption (3.7),

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(u)}{\Xi_{t_{j-1}}(u)}] = \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\exp(\sum_{k=1}^{K}\Big(iu(X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) + ((Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\psi_k(u))\Big))] = 1. \tag{3.9}$$

By differentiating equation (3.9) once and twice with respect to u, we obtain after simplification:

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(u)}{\Xi_{t_{j-1}}(u)}\sum_{k=1}^{K}\Big((X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) - m_k(iu)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\Big)] = 0,$$

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(u)}{\Xi_{t_{j-1}}(u)}\Big((\sum_{k=1}^{K}((X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) - m_k(iu)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})))^2 - \sum_{k=1}^{K}(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\psi''_k(u)\Big)] = 0. \tag{3.10}$$

We will use these equations later.

Since, given K Lévy processes $X_t^{(k)}$ and K time-change processes $Y_t^{(k)}$, one can compute the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$, for cases of interest, via conditioning arguments and by using results in Carr and Wu (2004) and Duffie et al. (2000), we will say nothing more about this and take $\Phi(z_1, z_2, z_3; j)$ as given. We refer the reader to Hong (2004) and Itkin and Carr (2010) who provide detailed computations and examples of the conditioning arguments for some common cases of interest.

In general, we allow, for each $k = 1, 2, \ldots, K$ and $\ell = 1, 2, \ldots, K$, $Y_t^{(k)}$ and $X_t^{(\ell)}$ to have non-zero covariance. However, a small number of results that we will derive are only valid when, for all $k = 1, 2, \ldots, K$ and for all $\ell = 1, 2, \ldots, K$, $Y_t^{(k)}$ and $X_t^{(\ell)}$ are independent. We shall indicate when this restriction is in force by referring to the following assumption.

Assumption 3.8. For all $k = 1, 2, \ldots, K$ and for all $\ell = 1, 2, \ldots, K$, $Y_t^{(k)}$ and $X_t^{(\ell)}$ are independent.

There is a further special case which will be of interest to us. We shall indicate when this restriction is in force by referring to the following assumption.

Assumption 3.9. The time-change processes are common in the sense that $Y_t^{(k)} = Y_t^{(1)}$, for all $k = 1, 2, \ldots, K$.

We stress that in the case of $Y_t^{(k)}$ being deterministic for all $k = 1, 2, \ldots, K$, then Assumption (3.8) is not an extra assumption. Furthermore, we stress that in the case that $K = 1$, then Assumption (3.9) is not an extra assumption as it must automatically hold.

4. Generalised variance swaps

In this section, our aim is to price generalised variance swaps using an extension of the methodology outlined in Section 1. We will use the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$ which we defined in equation (3.8).

We note that the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$ allows us to immediately evaluate the price of a discretely monitored proportional variance swap (defined in Section 2, equation (2.6)). We state the price in the next proposition.

Proposition 4.1. *The price* $\mathrm{PVS}(t_0, T, N)$*, at time* t_0*, of a (discretely monitored) proportional variance swap is*

$$\mathrm{PVS}(t_0, T, N) = P(t_0, T)\Big(\sum_{j=1}^{N} \big(D(t_{j-1}, t_j, 2)\Phi(0, -2i, 0; j) - 2D(t_{j-1}, t_j, 1) + 1\big)\Big). \tag{4.1}$$

Proof: We let $z_1 = 0$, $z_2 = -2i$ and $z_3 = 0$ in equation (3.8), then sum over j and simplify. •

We will examine the limit as $N \to \infty$ of equation (4.1) in Section 6.4.

Now we return to considering the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$ defined by equation (3.8). Differentiating equation (3.8) with respect to z_2 and dividing by i, we obtain

$$\begin{aligned}
\frac{1}{i}\frac{\partial \Phi(z_1, z_2, z_3; j)}{\partial z_2} &= \mathbb{E}_{t_0}^{\mathbb{Q}}[\log \frac{S(t_j)}{S(t_{j-1})} \exp(iz_1 \log \frac{S(t_{j-1})}{S(t_0)} + iz_2 \log \frac{S(t_j)}{S(t_{j-1})} + iz_3 \log \frac{S(t_N)}{S(t_j)})] \\
&= D(t_0, t_{j-1}, iz_1) D(t_{j-1}, t_j, iz_2) D(t_j, t_N, iz_3) \times \\
&\quad \mathbb{E}_{t_0}^{\mathbb{Q}}\Big[\exp(iz_1 \sum_{k=1}^{K} X_{Y_{t_{j-1}}}^{(k)}) \mathbb{E}_{t_{j-1}}^{\mathbb{Q}}\big[\frac{\Xi_{t_j}(z_2)}{\Xi_{t_{j-1}}(z_2)} \exp(-\sum_{k=1}^{K} (Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)})\psi_k(z_2)) \\
&\quad \times \Big(\varpi^{(j)}(iz_2) + \sum_{k=1}^{K} \big((X_{Y_{t_j}}^{(k)} - X_{Y_{t_{j-1}}}^{(k)}) - m_k(iz_2)(Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)})\big)\Big) \\
&\quad \times \mathbb{E}_{t_j}^{\mathbb{Q}}[\exp(iz_3 \sum_{k=1}^{K} (X_{Y_{t_N}}^{(k)} - X_{Y_{t_j}}^{(k)}))]\big]\Big], \quad \text{where}
\end{aligned} \tag{4.2}$$

$$\varpi^{(j)}(iu) \equiv \int_{t_{j-1}}^{t_j} (r(s) - q(s))ds + \sum_{k=1}^{K} m_k(iu)(Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)}). \tag{4.3}$$

It is now straightforward to value φ-log-entropy-forward-contracts, which we do in the next proposition.

Proposition 4.2. *The price* $\mathrm{LEFC}(t_0, T, \varphi)$*, at time* t_0*, of a* φ*-log-entropy-forward-contract is*

$$\mathrm{LEFC}(t_0, T, \varphi) \quad = \quad P(t_0, T) \sum_{k=1}^{K} m_k(\varphi) \mathbb{E}^{\mathbb{Q}}_{t_0}[\left(\frac{F(T,T)}{F(t_0,T)}\right)^{\varphi} \left(Y^{(k)}_T - Y^{(k)}_{t_0}\right)]. \tag{4.4}$$

Proof: We set $iz_1 = \varphi$, $iz_2 = \varphi$ and $iz_3 = \varphi$, for $\varphi = 0, 1$, in equation (4.2). We sum from $j = 1$ to N and then use equations (3.4), (3.10) and (4.3) to rearrange and simplify. •

We now proceed to value generalised variance swaps. We differentiate equation (4.2) again with respect to z_2 and again divide by i:

$$\begin{aligned}
-\frac{\partial^2 \Phi(z_1, z_2, z_3; j)}{\partial z_2^2} &= \mathbb{E}^{\mathbb{Q}}_{t_0}[(\log \frac{S(t_j)}{S(t_{j-1})})^2 \exp(iz_1 \log \frac{S(t_{j-1})}{S(t_0)} + iz_2 \log \frac{S(t_j)}{S(t_{j-1})} + iz_3 \log \frac{S(t_N)}{S(t_j)})] \\
&= D(t_0, t_{j-1}, iz_1) D(t_{j-1}, t_j, iz_2) D(t_j, t_N, iz_3) \times \\
&\quad \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\exp(iz_1 \sum_{k=1}^{K} X^{(k)}_{Y_{t_{j-1}}}) \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}\big[\frac{\Xi_{t_j}(z_2)}{\Xi_{t_{j-1}}(z_2)} \exp(-\sum_{k=1}^{K} (Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}}) \psi_k(z_2)) \\
&\quad \times \Big(\varpi^{(j)\,2}(iz_2) + \Big\{2\varpi^{(j)}(iz_2) \sum_{k=1}^{K} \Big((X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) - m_k(iz_2)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\Big)\Big\} \\
&\quad + \left(\sum_{k=1}^{K} \left((X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) - m_k(iz_2)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\right)\right)^2 \\
&\quad - \sum_{k=1}^{K} \psi''_k(z_2)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}}) \\
&\quad + \sum_{k=1}^{K} \psi''_k(z_2)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\Big) \mathbb{E}^{\mathbb{Q}}_{t_j}[\exp(iz_3 \sum_{k=1}^{K} (X^{(k)}_{Y_{t_N}} - X^{(k)}_{Y_{t_j}}))]\big]\Big].
\end{aligned} \tag{4.5}$$

We state the price of a generalised variance swap in the next proposition after defining some auxiliary notation. We define

$$\begin{aligned}
\Omega_1(\Upsilon_1, \Upsilon_2, \Upsilon_3, N) &\equiv \sum_{j=1}^{N} D(t_0, t_{j-1}, \Upsilon_1) D(t_{j-1}, t_j, \Upsilon_2) D(t_j, t_N, \Upsilon_3) \times \\
&\quad \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\exp(\Upsilon_1 \sum_{k=1}^{K} X^{(k)}_{Y_{t_{j-1}}}) \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}\big[\frac{\Xi_{t_j}(-i\Upsilon_2)}{\Xi_{t_{j-1}}(-i\Upsilon_2)} \exp(-\sum_{k=1}^{K} (Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}}) \psi_k(-i\Upsilon_2)) \\
&\quad \times \varpi^{(j)\,2}(\Upsilon_2) \mathbb{E}^{\mathbb{Q}}_{t_j}[\exp(\Upsilon_3 \sum_{k=1}^{K} (X^{(k)}_{Y_{t_N}} - X^{(k)}_{Y_{t_j}}))]\big]\Big],
\end{aligned}$$

$$\begin{aligned}
\Omega_2(\Upsilon_1, \Upsilon_2, \Upsilon_3, N) &\equiv \sum_{j=1}^{N} D(t_0, t_{j-1}, \Upsilon_1) D(t_{j-1}, t_j, \Upsilon_2) D(t_j, t_N, \Upsilon_3) \times \\
&\quad \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\exp(\Upsilon_1 \sum_{k=1}^{K} X^{(k)}_{Y_{t_{j-1}}}) \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}\big[\frac{\Xi_{t_j}(-i\Upsilon_2)}{\Xi_{t_{j-1}}(-i\Upsilon_2)} \exp(-\sum_{k=1}^{K} (Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}}) \psi_k(-i\Upsilon_2)) \\
&\quad \times 2\varpi^{(j)}(\Upsilon_2) \sum_{k=1}^{K} \Big((X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) - m_k(\Upsilon_2)(Y^{(k)}_{t_j} - Y^{(k)}_{t_{j-1}})\Big) \\
&\quad \mathbb{E}^{\mathbb{Q}}_{t_j}[\exp(\Upsilon_3 \sum_{k=1}^{K} (X^{(k)}_{Y_{t_N}} - X^{(k)}_{Y_{t_j}}))]\big]\Big],
\end{aligned}$$

$$
\begin{aligned}
\Omega_3(\Upsilon_1,\Upsilon_2,\Upsilon_3,N) \quad \equiv \quad & \sum_{j=1}^{N} D(t_0,t_{j-1},\Upsilon_1)D(t_{j-1},t_j,\Upsilon_2)D(t_j,t_N,\Upsilon_3)\times \\
& \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\exp(\Upsilon_1\sum_{k=1}^{K}X^{(k)}_{Y_{t_{j-1}}})\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}\big[\frac{\Xi_{t_j}(-i\Upsilon_2)}{\Xi_{t_{j-1}}(-i\Upsilon_2)}\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_j}-Y^{(k)}_{t_{j-1}})\psi_k(-i\Upsilon_2)) \\
& \times\Big(\Big(\sum_{k=1}^{K}((X^{(k)}_{Y_{t_j}}-X^{(k)}_{Y_{t_{j-1}}})-m_k(\Upsilon_2)(Y^{(k)}_{t_j}-Y^{(k)}_{t_{j-1}}))\Big)^2 \\
& -\sum_{k=1}^{K}\psi''_k(-i\Upsilon_2)(Y^{(k)}_{t_j}-Y^{(k)}_{t_{j-1}})\Big)\mathbb{E}^{\mathbb{Q}}_{t_j}[\exp(\Upsilon_3\sum_{k=1}^{K}(X^{(k)}_{Y_{t_N}}-X^{(k)}_{Y_{t_j}}))]\big]\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Omega_4(\Upsilon_1,\Upsilon_2,\Upsilon_3,N) \quad \equiv \quad & \sum_{j=1}^{N} D(t_0,t_{j-1},\Upsilon_1)D(t_{j-1},t_j,\Upsilon_2)D(t_j,t_N,\Upsilon_3)\times \\
& \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\exp(\Upsilon_1\sum_{k=1}^{K}X^{(k)}_{Y_{t_{j-1}}})\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}\big[\frac{\Xi_{t_j}(-i\Upsilon_2)}{\Xi_{t_{j-1}}(-i\Upsilon_2)}\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_j}-Y^{(k)}_{t_{j-1}})\psi_k(-i\Upsilon_2)) \\
& \times\sum_{k=1}^{K}\psi''_k(-i\Upsilon_2)(Y^{(k)}_{t_j}-Y^{(k)}_{t_{j-1}})\mathbb{E}^{\mathbb{Q}}_{t_j}[\exp(\Upsilon_3\sum_{k=1}^{K}(X^{(k)}_{Y_{t_N}}-X^{(k)}_{Y_{t_j}}))]\big]\Big].
\end{aligned}
$$

Proposition 4.3. *The price* $\mathrm{GVS}(t_0,T,N,\Upsilon_1,\Upsilon_2,\Upsilon_3)$*, at time* t_0*, of a generalised variance swap is*

$$
\begin{aligned}
\mathrm{GVS}(t_0,T,N,\Upsilon_1,\Upsilon_2,\Upsilon_3) \quad = \quad & P(t_0,T)\Big(\Omega_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)+\Omega_2(\Upsilon_1,\Upsilon_2,\Upsilon_3,N) \\
& +\Omega_3(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)+\Omega_4(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)\Big).
\end{aligned}
\tag{4.6}
$$

Proof: We set $iz_1=\Upsilon_1$, $iz_2=\Upsilon_2$ and $iz_3=\Upsilon_3$, for real Υ_1, Υ_2 and Υ_3, in equation (4.5) and multiply by the deterministic discount factor $P(t_0,T)\equiv\exp(-\int_{t_0}^{T}r(s)ds)$. Then we rearrange and sum over j. •

Note that, in order to get the price of a generalised variance swap we simply need to differentiate (either analytically (probably, using a symbolic mathematics computer package) or numerically by finite differences (perhaps, using Ridder's method to minimise the impact of numerical rounding errors)) the joint extended characteristic function twice as indicated on the first line of equation (4.6). The point of decomposing the price into the four components as on the final line of equation (4.6) is that it enables us to examine the four components individually in order to examine, for example, whether the terms are positive or negative, whether the terms vanish under the independence Assumption (3.8) or to what values the terms converge as $N\to\infty$.

Remark 4.4. Note that $\varpi^{(j)\,2}(\Upsilon_2)\geq 0$, since $\varpi^{(j)}(\Upsilon_2)$ is real. In fact, observing the form of $\varpi^{(j)}(\Upsilon_2)$ (see equation (4.3)), it is clear, except in the special case when the processes $Y^{(k)}_t$, for all $k=1,2,\ldots,K$, are deterministic (i.e. of type 1 of Assumption (3.4)) and furthermore there is a coincidental special combination of parameters, that $\varpi^{(j)}(\Upsilon_2)$ is not equal to zero and so $\varpi^{(j)\,2}(\Upsilon_2)$ would, in fact, except in this special case, be strictly positive. Clearly, $\Omega_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)$ is greater than or equal to zero, with equality only in the special case that $\varpi^{(j)}(\Upsilon_2)=0$ for all $j=1$ to N.

Remark 4.5. Note that the terms $\Omega_2(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)$ and $\Omega_3(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)$ both vanish under the independence Assumption (3.8), and therefore they will also both vanish if, for all $k=1,2,\ldots,K$, $Y^{(k)}_t$ is deterministic i.e. of type 1 of Assumption (3.4). Furthermore, the term $\Omega_3(\Upsilon_1,\Upsilon_2,\Upsilon_3,N)$ also vanishes if

Υ_2 is either 0 or 1 **and** Υ_3 is either 0 or 1. This can be seen from the second part of equation (3.10) after noting that $\psi_k(-i\Upsilon_2) = 0$, for all k, if Υ_2 is either 0 or 1.

Remark 4.6. Note that the term $\Omega_4(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ is strictly positive (using equation (3.4)).

5. Variance swaps

In the previous section, we considered generalised variance swaps. In this section, we specialise the results of the previous section to variance swaps. We show that, under stated conditions, discretely monitored variance swaps have higher prices than continuously monitored variance swaps and discuss the rate of convergence of the prices of discretely monitored variance swaps to their continuously monitored counterparts. We also provide analogous results for self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps.

The price of a variance swap $\mathrm{VS}(t_0, T, N) \equiv \mathrm{GVS}(t_0, T, N, 0, 0, 0)$ can be obtained as a special case of the price of a generalised variance swap. We state the result in the next proposition.

Proposition 5.1. *The price* $\mathrm{VS}(t_0, T, N)$*, at time* t_0*, of a variance swap is:*

$$\begin{aligned} \mathrm{VS}(t_0,T,N) &= P(t_0,T)\sum_{k=1}^{K}\psi_k''(0)\mathbb{E}^{\mathbb{Q}}_{t_0}\left[Y_T^{(k)} - Y_{t_0}^{(k)}\right] + P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[(\varpi^{(j)\,2}(0))]\Big] \\ &+ P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2\left(\sum_{k=1}^{K}\big(m_k(0)(Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)})\big)\right)A(t_{j-1},t_j)]\Big], \ \textit{where} \end{aligned}$$

$$A(t_{j-1},t_j) \equiv \sum_{k=1}^{K}\left((X^{(k)}_{Y_{t_j}} - X^{(k)}_{Y_{t_{j-1}}}) - m_k(0)(Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)})\right). \tag{5.1}$$

Proof: We let $\Upsilon_1 = 0$, $\Upsilon_2 = 0$ and $\Upsilon_3 = 0$ in equation (4.6), then use equation (3.10). •

We will see in Proposition 6.2 (see also Carr and Lee (2009)) that the first term in equation (5.1), namely $P(t_0,T)\sum_{k=1}^{K}\psi_k''(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T^{(k)} - Y_{t_0}^{(k)}]$, is the price, at time t_0, of a continuously monitored variance swap.

Remark 5.2. Equation (5.1) significantly extends Proposition 5.1 of Carr and Lee (2009) in that we allow K to be greater than one and we allow $Y_t^{(k)}$ and $X_t^{(k)}$ to be correlated or to be co-dependent. Note that the final term in equation (5.1) somewhat resembles the price of a derivative written on the covariance between the time-change processes and the log of the stock price (with a drift adjustment). In particular, this term vanishes under the independence Assumption (3.8). To explain this in more detail, note that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2\left(\sum_{k=1}^{K}\big(m_k(0)(Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)})\big)\right)A(t_{j-1},t_j)]\Big] = \\ &\mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\sum_{k=1}^{K}\mathrm{Covar}^{\mathbb{Q}}_{t_{j-1}}\big[2m_k(0)(Y_{t_j}^{(k)} - Y_{t_{j-1}}^{(k)}), \mathbb{E}^{\mathbb{Q}}[A(t_{j-1},t_j)|\mathcal{F}_{t_{j-1}} \vee \sigma\{Y_{t_j}^{(k)}, k = 1,2,\ldots,K\}]\big]\Big]. \end{aligned} \tag{5.2}$$

Clearly, the right-hand-side of the last equation vanishes if the processes $X_t^{(\ell)}$ and $Y_t^{(k)}$ are independent for all $\ell = 1, 2, \ldots, K$ and for all $k = 1, 2, \ldots, K$.

In the special case of the independence Assumption (3.8) and when $K = 1$, it can easily be verified by straightforward algebraic rearrangement that equation (5.1) is in agreement with Proposition 5.1 of Carr and Lee (2009) - although our method of proof is completely different.

When Assumption (3.8) is not in force, then it is straightforward, in cases of practical interest, to determine the sign of the term in the final two lines of equation (5.1), since it is directly related, via equation (5.2), to the covariance between the time-change processes $Y_t^{(k)}$ and the Lévy processes $X_t^{(\ell)}$. Indeed, since $m_k(0) < 0$ for all $k = 1, 2, \ldots, K$, the term is largest (i.e. most positive) when $X_t^{(\ell)}$ and $Y_t^{(k)}$ are negatively correlated, for each ℓ and each k. In practice, stock price returns (under $\mathbb{Q}$) are nearly always negatively skewed and, hence, one would expect it be more likely that $X_t^{(\ell)}$ and $Y_t^{(k)}$ would be negatively correlated. Of course, this is in no way guaranteed, since the negative skewness could come from, for example, $X_t^{(\ell)}$ and $Y_t^{(k)}$ having zero correlation or even a small positive correlation but with the Lévy processes $X_t^{(\ell)}$ being highly negatively skewed. Nevertheless, on balance, (see Carr et al. (2003) for more formal empirical evidence), it seems more likely that $X_t^{(\ell)}$ and $Y_t^{(k)}$ would be negatively correlated. When this is the case, for each ℓ and each k, equation (5.1) shows that, other things being equal, the price of a discretely monitored variance swap is higher than the price of the same discretely monitored variance swap under the independence Assumption (3.8).

Remark 5.3. Note that, since $\varpi^{(j)}(0)$ is the drift term for the log of the stock price under $\mathbb{Q}$ over the time period t_{j-1} to t_j, the term $\varpi^{(j)\,2}(0)$ is just the square of the drift and is clearly (see Remark 4.4) non-negative. Therefore, as also shown by Carr and Lee (2009), under the independence Assumption (3.8), a discretely monitored variance swap is always worth at least as much as an otherwise identical continuously monitored variance swap (with equality only in the special case that $\varpi^{(j)}(0) \equiv 0$, for all j). Clearly, Remark 5.2 extends both this remark and Carr and Lee (2009) when the independence assumption is dropped.

6. Convergence as $N \to \infty$

In this section, we consider the limit as $N \to \infty$ of the variance-related swaps defined in Section 2. In the first sub-section, we consider variance swaps. In subsequent sub-sections, we consider the limits for self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps.

In order to prove convergence results in the limit as $N \to \infty$, we need to make some extra assumptions.

Assumption 6.1. For each $k = 1, 2, \ldots, K$, $Y_t^{(k)}$ is one of the following types:

- 1. Either $Y_t^{(k)}$ is an increasing, **continuous,** deterministic function of t, $t \in [t_0, \infty)$.
- 2. Or $Y_t^{(k)}$ is an increasing, **continuous** stochastic process. In general, it may be correlated with $X_t^{(\ell)}$, for any $\ell = 1, 2, \ldots, K$.

Note that $Y_t^{(k)}$ need not necessarily be of the same type for all k.

For future reference, we introduce the following notation: We define the probability measure $\mathbb{Q}\,(1)$, equivalent to the measure $\mathbb{Q}$, under which $S(t)$ is the numeraire. In order to collectively refer to both $\mathbb{Q}$ and $\mathbb{Q}\,(1)$, we let $\varphi = 0, 1$, and define $\mathbb{Q}\,(\varphi)$ to be such that $\mathbb{Q}\,(\varphi)$ is the same as $\mathbb{Q}$ if $\varphi = 0$ and $\mathbb{Q}\,(\varphi)$ is the same as $\mathbb{Q}\,(1)$ if $\varphi = 1$. This will ease notation in the sequel.

In order to prove convergence results (i.e. Propositions 6.2, 6.3, 6.5 and 6.6), we, firstly, need to define a sequence of partitions of the interval $[t_0, T]$. We define $\Pi_n = \{t_{0,(n)}, t_{1,(n)}, \ldots, t_{K(n),(n)}\}$, where $n \geq 1$, and where for all n, $t_{0,(n)} = t_0 \equiv 0$, $t_{K(n),(n)} = T$. We define $\Delta_n \equiv \sup\{t_{i,(n)} - t_{i-1,(n)} : i = 1, 2, \ldots, n\}$. We assume that $\lim_{n\to\infty} \Delta_n = 0$. More specifically, Δ_n is $O(1/n)$ as $n \to \infty$. Secondly, we also need to make further assumptions, mainly on the time-change processes $Y_t^{(k)}$, which we specify shortly (see equations (6.3), (6.4), (6.12), (6.13) and (6.17)). In making these assumptions, we use big-O notation. We define this by saying that a stochastic process Z_n is $O(f(n))$ if there exists a random variable $L(\omega)$, $\omega \in \Omega$, such that $|Z_n/f(n)| < L(\omega)$ almost surely.

6.1. **Convergence of variance swaps.** We define a continuously monitored variance swap to have a payoff at time T equal to the limit as $N \to \infty$ of that given in Section 2:

$$\begin{aligned} \lim_{N\to\infty} \sum_{j=1}^{N} \left(\log \frac{S(t_j)}{S(t_{j-1})}\right)^2 &= \sum_{k=1}^{K} \Big([X^{c,(k)}_{Y_\bullet}, X^{c,(k)}_{Y_\bullet}]_T + \sum_{t_0<u\le T} (\Delta X^{d,(k)}_{Y_u})^2\Big) \\ &= \sum_{k=1}^{K} \Big(\sigma^{(k)\,2}(Y^{(k)}_T - Y^{(k)}_{t_0}) + \sum_{t_0<u\le T} (\Delta X^{d,(k)}_{Y_u})^2\Big), \end{aligned} \tag{6.1}$$

where square brackets denote quadratic variation and the right-hand-side of the first line is a well-known result to which the second line is equivalent (see, for example, Section 2 of Carr and Lee (2010b)). It follows, from directly computing the expectation under $\mathbb{Q}$ or from, for example, Carr and Lee (2009), that the price $\mathrm{VS}(t_0, T, \infty)$, at time t_0, of the continuously monitored variance swap is

$$\begin{aligned} & P(t_0,T) \sum_{k=1}^{K} \Big(\sigma^{(k)\,2} + \int_{-\infty}^{\infty} x^2 \nu^{(k)}(dx)\Big) \mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}] \\ &= P(t_0,T) \sum_{k=1}^{K} \psi_k^{''}(0) \mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}]. \end{aligned} \tag{6.2}$$

This pricing result is well-known - the point is that the price in equation (6.2) is the same as that which we shall obtain in the first line of equation (6.5) in Proposition 6.2 below by taking the limit $N \to \infty$ of $\mathrm{VS}(t_0, T, N)$. This will justify the notation $\mathrm{VS}(t_0, T, \infty)$ by confirming it is the same as $\lim_{N\to\infty} \mathrm{VS}(t_0, T, N)$. We now turn to Proposition 6.2.

Proposition 6.2. *Assume that the following regularity conditions hold:*
For each $k = 1, 2, \dots, K$, $\ell = 1, 2, \dots, K$ and for each $j = 1, \dots, n$, as $n \to \infty$:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}\,(\varphi)}_{t_{j-1,(n)}}[(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})] &= O(\Delta_n) \ a.s., \\ \mathbb{E}^{\mathbb{Q}\,(\varphi)}_{t_{j-1,(n)}}[(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})(Y^{(\ell)}_{t_{j,(n)}} - Y^{(\ell)}_{t_{j-1,(n)}})] &= O(\Delta_n^2) \ a.s., \end{aligned} \tag{6.3}$$

$$\begin{aligned} & \mathbb{V}ar^{\mathbb{Q}\,(\varphi)}_{t_{j-1,(n)}}[\mathbb{E}^{\mathbb{Q}\,(\varphi)}[\sum_{k=1}^{K} \Big((X^{(k)}_{Y_{t_{j,(n)}}} - X^{(k)}_{Y_{t_{j-1,(n)}}}) - m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\Big) \mid \\ & \mathcal{F}_{t_{j-1}} \vee \sigma\{Y^{(k)}_{t_j}, k = 1, 2, \dots, K\}]] = O(\Delta_n^2) \ a.s. \end{aligned} \tag{6.4}$$

Then:

$$\begin{aligned} \lim_{N\to\infty} \mathrm{VS}(t_0,T,N) &= P(t_0,T) \sum_{k=1}^{K} \psi_k^{''}(0) \mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}] = P(t_0,T)\Omega_4(0,0,0,\infty) \\ &= \mathrm{VS}(t_0,T,\infty), \end{aligned} \tag{6.5}$$

$$\textit{and} \qquad |\mathrm{VS}(t_0,T,N) - \lim_{N\to\infty} \mathrm{VS}(t_0,T,N)| = O(\Delta_N) \ \textit{as } N \to \infty.$$

Proof: See the Appendix. •

We see that $\mathrm{VS}(t_0, T, \infty) = \lim_{N\to\infty} \mathrm{VS}(t_0, T, N)$ only depends on the Lévy processes via $\psi_k^{''}(0)$ and only depends on the time-change processes via $\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}]$.

6.2. **Self-quantoed variance swaps and gamma swaps.** There are results for self-quantoed variance swaps (respectively, gamma swaps) which are analogous to those for variance swaps. We put $\Upsilon_1 = 1$, $\Upsilon_2 = 1$ and $\Upsilon_3 = 1$ (respectively, $\Upsilon_1 = 0$, $\Upsilon_2 = 1$ and $\Upsilon_3 = 0$) in our formulae for generalised variance swaps (equation (4.6)). This immediately gives us the price $\mathrm{GVS}(t_0, T, N, 1, 1, 1)$ (respectively, $\mathrm{GVS}(t_0, T, N, 0, 1, 0)$), at time t_0, of a discretely monitored self-quantoed variance swap (respectively, gamma swap). We consider the limit as $N \to \infty$ of $\mathrm{GVS}(t_0, T, N, 1, 1, 1)$ (respectively, $\mathrm{GVS}(t_0, T, N, 0, 1, 0)$) in the next proposition. Before that, we define continuously monitored self-quantoed variance swaps and gamma swaps to have payoffs at time T respectively equal to the limits as $N \to \infty$ of those given in Section 2:

$$\lim_{N\to\infty} \sum_{j=1}^{N} \frac{S(T)}{S(t_0)} \left(\log \frac{S(t_j)}{S(t_{j-1})} \right)^2 = \frac{S(T)}{S(t_0)} \sum_{k=1}^{K} \Big([X^{c,(k)}_{Y_\bullet}, X^{c,(k)}_{Y_\bullet}]_T + \sum_{t_0 < u \leq T} (\Delta X^{d,(k)}_{Y_u})^2 \Big),$$

$$\lim_{N\to\infty} \sum_{j=1}^{N} \frac{S(t_j)}{S(t_{j-1})} \left(\log \frac{S(t_j)}{S(t_{j-1})} \right)^2 = \sum_{k=1}^{K} \Big([X^{c,(k)}_{Y_\bullet}, X^{c,(k)}_{Y_\bullet}]_T + \sum_{t_0 < u \leq T} \exp(\Delta X^{d,(k)}_{Y_u}) (\Delta X^{d,(k)}_{Y_u})^2 \Big),$$

where we have used Proposition 2.3 of Carr and Lee (2010b) to justify the right-hand-sides. It follows, by directly taking expectations under $\mathbb{Q}$ and using Proposition 3.2 of Carr and Lee (2010b), that the prices $\mathrm{GVS}(t_0, T, \infty, 1, 1, 1)$ and $\mathrm{GVS}(t_0, T, \infty, 0, 1, 0)$, at time t_0, of the continuously monitored self-quantoed variance swaps and gamma swaps are respectively given by the right-hand-sides of equations (6.6) and (6.7) below. Again, as in Sub-Section 6.1, the point is that this will justify the notation $\mathrm{GVS}(t_0, T, \infty, 1, 1, 1)$ and $\mathrm{GVS}(t_0, T, \infty, 0, 1, 0)$ by confirming that they are the same as $\lim_{N\to\infty} \mathrm{GVS}(t_0, T, N, 1, 1, 1)$ and $\lim_{N\to\infty} \mathrm{GVS}(t_0, T, N, 0, 1, 0)$ respectively - we will evaluate these limits in Proposition 6.3 below.

Proposition 6.3. *Under the regularity conditions in equations* (6.3) *and* (6.4)*,*

$$\lim_{N\to\infty} \mathrm{GVS}(t_0, T, N, 1, 1, 1) = P(t_0, T) \sum_{k=1}^{K} \psi_k^{''}(-i) \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\frac{F(T,T)}{F(t_0,T)} (Y^{(k)}_T - Y^{(k)}_{t_0})\Big], \tag{6.6}$$

$$\lim_{N\to\infty} \mathrm{GVS}(t_0, T, N, 0, 1, 0) = P(t_0, T) \sum_{k=1}^{K} \psi_k^{''}(-i) \mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}], \tag{6.7}$$

and
$$|\mathrm{GVS}(t_0, T, N, 1, 1, 1) - \lim_{N\to\infty} \mathrm{GVS}(t_0, T, N, 1, 1, 1)| = O(\Delta_N) \text{ as } N \to \infty,$$
$$|\mathrm{GVS}(t_0, T, N, 0, 1, 0) - \lim_{N\to\infty} \mathrm{GVS}(t_0, T, N, 0, 1, 0)| = O(\Delta_N) \text{ as } N \to \infty.$$

Proof: See the Appendix. •

We note that for variance swaps, self-quantoed variance swaps and gamma swaps, the $\Omega_4(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term (with the relevant values of Υ_1, Υ_2, Υ_3) is the price, at time t_0, of the continuously monitored version of the swap while the $\Omega_3(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term is identically zero. The $\Omega_1(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term is a non-negative term which is loosely speaking (in the sense of Remark 4.4 and in the sense that the term vanishes in the limit $N \to \infty$) related to discrete monitoring, while the $\Omega_2(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term is a term that vanishes under the independence Assumption (3.8). We have related the sign of this term, in the case of variance swaps, to the sign of the correlation between $X^{(\ell)}_t$ and $Y^{(k)}_t$, for each ℓ and each k, in Remark 5.2. For self-quantoed variance swaps and gamma swaps, since $m_k(1) > 0$, the sign of this term is reversed, i.e. the $\Omega_2(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term is largest (i.e. most positive) when $X^{(\ell)}_t$ and $Y^{(k)}_t$ are positively correlated, for each ℓ and each k.

Remark 6.4. In this remark, we consider the convergence of the price of a discretely monitored variance swap (respectively, self-quantoed variance swap, gamma swap) in the following special case (and we stress that this special case only applies within the confines of this remark):

We assume that there are no time-changes in the sense that $Y_t^{(k)} = t$, for all t and for all $k = 1, 2, \dots, K$. We assume that we have a discretely monitored variance swap (respectively, self-quantoed variance swap, gamma swap) with N equally spaced monitoring time ie the monitoring times are of the form: $t_i = t_0 + iT/N \equiv t_0 + i\Delta$, for all $i = 1, 2, \dots, N$.

Finally, we assume that either (a) $r(s) = q(s)$, for all $s \in [t_0, T]$ (which would always hold if the underlying process $S(t)$ is that of a futures price rather than a stock price, in the sense that the martingale property of futures prices under $\mathbb{Q}$ then implies that we can effectively always set $q(s) = r(s)$), or (b) $r(s)$ and $q(s)$ are constants.

So as to be able to cover both case (a) and (b) for either a variance swap, a self-quantoed variance swap or a gamma swap in one set of equations, we let μ be a constant which is defined as $\mu \equiv (r(s)-q(s))+\sum_{k=1}^{K} m_k(0)$ for the case of a variance swap and $\mu \equiv (r(s) - q(s)) + \sum_{k=1}^{K} m_k(1)$ for the case of a self-quantoed variance swap or a gamma swap.

Since the $\Omega_2(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term and the $\Omega_3(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term are identically zero, the price of the discretely monitored variance swap (respectively, self-quantoed variance swap, gamma swap) minus the price of the continuously monitored version of the same swap is the relevant $\Omega_1(\Upsilon_1, \Upsilon_2, \Upsilon_3, N)$ term which, in all three cases is

$$\sum_{j=1}^{N} \left(\mu\Delta\right)^2 = \mu^2\Delta^2 N = \mu^2 T^2/N. \tag{6.8}$$

We therefore see that, in this special case, the convergence rate is exactly $1/N$. In other words, **the asymptotic convergence rate seen in Propositions 6.2 and 6.3 is exact in the special case contained within this remark.**

Furthermore, consider two variance swaps (respectively, self-quantoed variance swaps, gamma swaps) both with maturity at time T. We take N even. The first has N monitoring times, $0 \equiv t_0 < t_1 < ... < t_{j-1} < t_j < ... < t_N \equiv T$, and the second has $N/2$ monitoring times, $0 \equiv t_0 < t_2 < ... < t_{\bar{j}-2} < t_{\bar{j}} < ... < t_N \equiv T$, for even $\bar{j}$. The price of the second variance swap (respectively, self-quantoed variance swap, gamma swap) minus the price of the first variance swap (respectively, self-quantoed variance swap, gamma swap) is

$$\mu^2 T^2/(N/2) - \mu^2 T^2/N = \mu^2 T^2/N. \tag{6.9}$$

We can conclude that the price of the second variance swap (respectively, self-quantoed variance swap, gamma swap) is certainly at least that of the first variance swap (respectively, self-quantoed variance swap, gamma swap), i.e. **in the special case contained within this remark, the price of a variance swap (respectively, self-quantoed variance swap, gamma swap) is certainly non-increasing (and actually decreasing unless $\mu = 0$) as the number of monitoring times is doubled.**

6.3. **Skewness swaps.** We now consider skewness swaps which we defined in Section 2.

Consider a discretely monitored skewness swap. It has a price $\mathrm{SKS}(t_0,T,N)$, at time t_0,

$$\begin{aligned}\mathrm{SKS}(t_0,T,N) &= P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}\big(\log\frac{S(t_j)}{S(t_{j-1})}\big)^3] \\ &= iP(t_0,T)\sum_{j=1}^{N}\frac{\partial^3\Phi(z_1,z_2,z_3;j)}{\partial z_2^3}|_{z_1=0,z_2=0,z_3=0}. \end{aligned} \tag{6.10}$$

We can easily compute an explicit formula for $\mathrm{SKS}(t_0,T,N)$ by differentiating equation (4.5) with respect to z_2 but we omit this for brevity.

We define the payoff of a continuously monitored skewness swap, at time T, to be equal to the limit as $N\to\infty$ of that given in Section 2:

$$\lim_{N\to\infty}\sum_{j=1}^{N}\left(\log\frac{S(t_j)}{S(t_{j-1})}\right)^3 = \sum_{t_0<u\le T}(\Delta X^{d,(k)}_{Y_u})^3, \tag{6.11}$$

using Proposition 2.3 of Carr and Lee (2010b) to justify the right-hand-side. It follows, by directly taking expectations under $\mathbb{Q}$ and using Proposition 3.2 of Carr and Lee (2010b), that the price $\mathrm{SKS}(t_0,T,\infty)$, at time t_0, of a continuously monitored skewness swap is $\mathrm{SKS}(t_0,T,\infty)=P(t_0,T)\sum_{k=1}^{K}\int_{-\infty}^{\infty}x^3\nu^{(k)}(dx)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T-Y^{(k)}_{t_0}]$, which is the same as in the first line of equation (6.14) in Proposition 6.5 below, which we will obtain by taking the limit as $N\to\infty$ of $\mathrm{SKS}(t_0,T,N)$.

We now turn to Proposition 6.5.

Proposition 6.5. *If the regularity conditions in equations* (6.3) *and* (6.4) *hold and also for each* $j=1,\dots,n$

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)^3]=O(\Delta_n^3)\ \ a.s., \quad and \tag{6.12}$$

$$\begin{aligned}&\mathbb{V}ar^{\mathbb{Q}}_{t_{j-1,(n)}}[\mathbb{E}^{\mathbb{Q}}[\big(\sum_{k=1}^{K}\Big((X^{(k)}_{Y_{t_{j,(n)}}}-X^{(k)}_{Y_{t_{j-1,(n)}}})-m_k(0)(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})\Big)\big)^2 \\ &|\mathcal{F}_{t_{j-1}}\vee\sigma\{Y^{(k)}_{t_j},k=1,2,\dots,K\}]]=O(\Delta_n^2)\ \ a.s.,\end{aligned} \tag{6.13}$$

then

$$\begin{aligned}\lim_{N\to\infty}\mathrm{SKS}(t_0,T,N) &= -iP(t_0,T)\sum_{k=1}^{K}\psi_k'''(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T-Y^{(k)}_{t_0}] \\ &= \mathrm{SKS}(t_0,T,\infty),\end{aligned} \tag{6.14}$$

$$and \qquad |\mathrm{SKS}(t_0,T,N)-\lim_{N\to\infty}\mathrm{SKS}(t_0,T,N)|=O(\Delta_N)\ \ as\ N\to\infty. \tag{6.15}$$

Proof: See the Appendix. •

6.4. **Proportional variance swaps.** We defined and priced discretely monitored proportional variance swaps in equation (4.1). For completeness, we also consider continuously monitored proportional variance swaps. We define a continuously monitored proportional variance swap to have a payoff at time T equal to the limit as $N\to\infty$ of that in equation (2.6):

$$\lim_{N\to\infty}\sum_{j=1}^{N}\big(\frac{S(t_j)}{S(t_{j-1})}-1\big)^2 = \sum_{k=1}^{K}\Big([X^{c,(k)}_{Y_\bullet},X^{c,(k)}_{Y_\bullet}]_T+\sum_{t_0<u\le T}(\exp(\Delta X^{d,(k)}_{Y_u})-1)^2\Big), \tag{6.16}$$

where we have again used Proposition 2.3 of Carr and Lee (2010b) to justify the right-hand-side. It follows, by directly taking expectations under $\mathbb{Q}$ and using Proposition 3.2 of Carr and Lee (2010b), or results in chapter 19 of Sato (1999), that the price $\mathrm{PVS}(t_0, T, \infty)$, at time t_0, of the continuously monitored proportional variance swap is the same as that given by the right-hand-side of the first line of equation (6.18) in Proposition 6.6 below, which we will obtain by considering the limit $N \to \infty$ of $\mathrm{PVS}(t_0, T, N)$.

Again, as in Sub-Section 6.1, the point is that this will justify the notation $\mathrm{PVS}(t_0, T, \infty)$ by confirming that it is the same as $\lim_{N\to\infty} \mathrm{PVS}(t_0, T, N)$.

We now turn to Proposition 6.6.

Proposition 6.6. *If the following regularity condition holds as $n \to \infty$:*

$$\begin{aligned} \Big|\mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}\big(\exp\big(-\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i)\big) - 1\big)\Big] - \\ - \sum_{k=1}^{K}\psi_k(-2i)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}}]\Big| = O(\Delta_n^2) \ \ a.s. \ , \end{aligned} \tag{6.17}$$

then:

$$\begin{aligned} \lim_{N\to\infty}\mathrm{PVS}(t_0, T, N) &= -P(t_0, T)\sum_{k=1}^{K}\psi_k(-2i)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}] \\ &= \mathrm{PVS}(t_0, T, \infty), \end{aligned} \tag{6.18}$$

$$\text{and} \qquad |\mathrm{PVS}(t_0, T, N) - \lim_{N\to\infty}\mathrm{PVS}(t_0, T, N)| = O(\Delta_N) \ \ as \ N \to \infty. \tag{6.19}$$

Proof: See the Appendix. •

Remark 6.7. It is clear from equation (4.1) (and using the result that $\psi_k(-2i) < 0$) that when $X^{(\ell)}_t$ and $Y^{(k)}_t$ are positively correlated, for each ℓ and k, the price of a discretely monitored proportional variance swap is higher than that of the same discretely monitored proportional variance swap under the independence Assumption (3.8). When $r(s) = q(s)$, for all $s \in [t_0, T]$, and under the independence Assumption (3.8), a discretely monitored proportional variance swap is always worth at least as much as an otherwise identical continuously monitored proportional variance swap. The proof of this is contained within the proof of Proposition 6.6. **This remark, in connection with proportional variance swaps, is analogous to Remark** 5.2 **for variance swaps (and hence also generalises Proposition 5.1 of Carr and Lee (2009)).**

We see that, as with variance swaps, self-quantoed variance swaps and gamma swaps, we can derive simple formulae for the prices of both discretely monitored and continuously monitored skewness swaps and proportional variance swaps, via the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$, in terms of the characteristic exponents $\psi_k(u)$ of the Lévy processes $X^{(k)}_t$.

We have not (for brevity) provided the explicit formulae for discretely monitored self-quantoed variance swaps, gamma swaps and skewness swaps but we stress again that, if one simply wishes to compute prices, one simply needs to differentiate the joint extended characteristic function twice or three times (perhaps, using a symbolic mathematics computer package, or numerically by finite differences (perhaps, using Ridder's method to minimise the impact of numerical rounding errors)).

We can see from equations (4.6), (5.1), (6.5), (6.7), (6.14) and (6.18), that the prices of continuously monitored variance swaps, continuously monitored gamma swaps, continuously monitored skewness swaps and continuously monitored proportional variance swaps swaps do not depend upon the correlation structure between $X_t^{(k)}$ and $Y_t^{(k)}$, whereas the prices of their discretely monitored counterparts do. This is noteworthy because there are typically two ways to introduce skew (asymmetry) into the implied volatility structure of a model. One way is to introduce a correlation (or co-dependency) between the Lévy processes $X_t^{(k)}$ and the stochastic time-change processes $Y_t^{(k)}$ (such as having a non-zero correlation in the Heston (1993) model). A second way is to have an uncorrelated time-change but to have a skewed Lévy process (eg. the VG (see Madan et al. (1998)) or CGMY (see Carr et al. (2003)) models with the M and G parameters different) (a third way is to combine the two approaches, but we leave this aside for now). The two different ways are contrasting. In the first way, changing the correlation structure will not change the prices of continuously monitored versions of the swaps listed above (though it will change those of their discretely monitored counterparts and of vanilla options). In the second way, changing the parameters M and G in the VG or CGMY models will change the prices of both continuously monitored and discretely monitored versions of the swaps listed above (and also of vanilla options).

In order to prove convergence results concerning the limit as $N \to \infty$, we have had to make some additional assumptions (eg. equations (6.3), (6.4) and (6.17)). These assumptions can be tested if the relevant expressions are known in closed form (typically by differentiating (in general, a symbolic mathematics computer package becomes essential) the joint extended characteristic function $\Phi(z_1, z_2, z_3; j)$ or the Laplace transform of $Y_t^{(k)}$, when these are known analytically). The following proposition shows that the assumptions in equations (6.3), (6.4) and (6.17) automatically hold when the time-change processes $Y_t^{(k)}$ are either deterministic or are integrated non-Gaussian OU processes (Barndorff-Nielsen and Shephard (2001)) or are integrated independent Heston (1993) processes, or combinations of the three.

Proposition 6.8. *Suppose that $Y_t^{(k)}$, for each k, is of the form*

$$Y_t^{(k)} = \int_{t_0}^{t} y_s^{(k)} ds \quad \text{where} \quad y_s^{(k)} \equiv g_k(s) + \sum_{m=1}^{M} \alpha_{mk} w_s^{(m)} \sum_{m=1}^{M} \beta_{mk} x_s^{(m)},$$

where, for each $m = 1, \ldots, M$, $w_t^{(m)}$ and $x_t^{(m)}$ follow under $\mathbb{Q}$

$$\begin{aligned} dw_t^{(m)} &= -\theta^{(m)} w_t^{(m)} dt + dN_t^{(m)}, \quad w_{t_0}^{(m)} \equiv w_0^{(m)}, \ (w_0^{(m)} > 0), \\ dx_t^{(m)} &= \kappa^{(m)} (\eta^{(m)} - x_t^{(m)}) dt + \lambda^{(m)} \sqrt{x_t^{(m)}} dz_t^{(m)}, \quad x_{t_0}^{(m)} \equiv x_0^{(m)}, \ (x_0^{(m)} > 0), \end{aligned} \tag{6.20}$$

*where $\theta^{(m)} > 0$, $\kappa^{(m)} > 0$, $\eta^{(m)} > 0$ and $\lambda^{(m)} \geq 0$ are (finite) constants and, for each m, $N_t^{(m)}$ is a compound Poisson processes with exponentially distributed (assumed always positive) jump amplitudes and $z_t^{(m)}$ is a standard Brownian motion. Furthermore, for each k, $g_k(t)$ is an increasing, non-negative continuous, differentiable deterministic function of t and, for each m and each k, α_{mk} and β_{mk} are non-negative constants. Assume for $\ell \neq m$ that $N_t^{(m)}$ and $N_t^{(\ell)}$ and $z_t^{(m)}$ and $z_t^{(\ell)}$ are independent (but, we do **NOT** necessarily assume that $N_t^{(\ell)}$ or $z_t^{(\ell)}$ are independent of the Lévy processes $X_t^{(k)}$, for $k = 1, \ldots, K$). Then equations* (6.3)*,* (6.4) *and* (6.17) *hold.*

Proof: See the Appendix. •

Note that the time-change processes $Y_t^{(k)}$ in Proposition 6.8 allows us to nest all the models referred to in Remark 3.6.

Broadie and Jain (2008a) show that, if the log of the underlying stock price follows a Merton (1976) jump-diffusion process or a Heston (1993) process, then as as the number N of monitoring times is allowed to tend to infinity, the prices of discretely monitored variance swaps exhibit $O(1/N)$ convergence to those of their continuously monitored counterparts. **We have significantly extended Broadie and Jain (2008a) by giving convergence results for self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps, under much more general time-changed Lévy process dynamics.**

We also remark that the undiscounted prices (equivalently, the fair strikes) of continuously monitored skewness swaps and continuously monitored proportional variance swaps, as with continuously monitored variance swaps, do not depend upon interest-rates or dividend yields.

Carr and Lee (2009) show that continuously monitored variance swaps written on the forward price and continuously monitored variance swaps written on the spot price have the same payoff and the same price when interest-rates and dividend yields are deterministic. Our analysis clearly shows that the same is also true for continuously monitored skewness swaps and continuously monitored proportional variance swaps.

7. Log-forward-contract and entropy-forward-contract multipliers

In this section, we show the relationship between continuously monitored variance swaps and log-forward-contracts. This relationship is known, thanks to Carr and Lee (2009), but our method of proof is novel and adds new insights. In addition, we are able to extend the results to the other types of swaps considered here. We do so in the following theorem:

Theorem 7.1. *The ratio of the price of a continuously monitored variance swap (respectively, a self-quantoed variance swap, a gamma swap, a skewness swap and a proportional variance swap) to the price of a log-forward-contract (respectively, an entropy-forward-contract, a log-forward-contract, a log-forward-contract and a log-forward-contract) is*

$$\frac{\mathrm{VS}(t_0,T,\infty)}{LEFC(t_0,T,0)} \equiv -Q_X^{\mathrm{VS}}, \quad \textit{where} \quad Q_X^{\mathrm{VS}} = \frac{\sum_{k=1}^{K}\psi_k''(0)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]}{\sum_{k=1}^{K}|m_k(0)|\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]} > 0. \tag{7.1}$$

$$\frac{\mathrm{GVS}(t_0,T,\infty,1,1,1)}{LEFC(t_0,T,1)} \equiv Q_X^{\mathrm{SQS}},\ Q_X^{\mathrm{SQS}} = \frac{\sum_{k=1}^{K}\psi_k''(-i)\mathbb{E}_{t_0}^{\mathbb{Q}}[\frac{F(T,T)}{F(t_0,T)}(Y_T^{(k)}-Y_{t_0}^{(k)})]}{\sum_{k=1}^{K}m_k(1)\mathbb{E}_{t_0}^{\mathbb{Q}}[\frac{F(T,T)}{F(t_0,T)}(Y_T^{(k)}-Y_{t_0}^{(k)})]} > 0. \tag{7.2}$$

$$\frac{\mathrm{GVS}(t_0,T,\infty,0,1,0)}{LEFC(t_0,T,0)} \equiv -Q_X^{\mathrm{GS}}, \quad Q_X^{\mathrm{GS}} = \frac{\sum_{k=1}^{K}\psi_k''(-i)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]}{\sum_{k=1}^{K}|m_k(0)|\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]} > 0. \tag{7.3}$$

$$\frac{\mathrm{SKS}(t_0,T,\infty)}{LEFC(t_0,T,0)} \equiv -Q_X^{\mathrm{SKS}}, \quad Q_X^{\mathrm{SKS}} = \frac{-i\sum_{k=1}^{K}\psi_k'''(0)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]}{\sum_{k=1}^{K}|m_k(0)|\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]}. \tag{7.4}$$

$$\frac{\mathrm{PVS}(t_0,T,\infty)}{LEFC(t_0,T,0)} \equiv -Q_X^{\mathrm{PVS}}, \quad Q_X^{\mathrm{PVS}} = \frac{-\sum_{k=1}^{K}\psi_k(-2i)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]}{\sum_{k=1}^{K}|m_k(0)|\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)}-Y_{t_0}^{(k)}]} > 0. \tag{7.5}$$

Proof: Divide equations (6.5) (respectively, equations (6.6), (6.7), (6.14) and (6.18)) by equation (4.4) with $\varphi = 0$ (respectively, $\varphi = 1$, $\varphi = 0$, $\varphi = 0$ and $\varphi = 0$). •

Following Carr and Lee (2009), we refer to Q_X^{VS}, Q_X^{SQS}, Q_X^{GS}, Q_X^{SKS} and Q_X^{PVS}, defined in the previous equations, as multipliers. Theorem 7.1 leads to three corollaries.

Corollary 7.2. *In the special case of Theorem* 7.1 *when Assumption* (3.9) *holds (i.e. there is a common time-change - which must automatically be true when $K = 1$), then*

$$Q_X^{\mathrm{VS}} = \frac{\sum_{k=1}^K \psi_k^{''}(0)}{\sum_{k=1}^K |m_k(0)|},\ Q_X^{\mathrm{SQS}} = \frac{\sum_{k=1}^K \psi_k^{''}(-i)}{\sum_{k=1}^K m_k(1)},$$
$$Q_X^{\mathrm{GS}} = \frac{\sum_{k=1}^K \psi_k^{''}(-i)}{\sum_{k=1}^K |m_k(0)|},\ Q_X^{\mathrm{SKS}} = \frac{-i\sum_{k=1}^K \psi_k^{'''}(0)}{\sum_{k=1}^K |m_k(0)|},\ Q_X^{\mathrm{PVS}} = \frac{-\sum_{k=1}^K \psi_k(-2i)}{\sum_{k=1}^K |m_k(0)|}.$$

Proof: Trivially, the terms involving $\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)} - Y_{t_0}^{(k)}]$ or $\mathbb{E}_{t_0}^{\mathbb{Q}}[\frac{F(T,T)}{F(t_0,T)}(Y_T^{(k)} - Y_{t_0}^{(k)})]$ cancel. $\bullet$

The first part of Corollary 7.2 (i.e. for Q_X^{VS}) is proven in Carr and Lee (2009) by completely different methods.

Remark 7.3. Note that, when Assumption (3.9) holds (which it will in many models of interest), the multipliers Q_X^{VS}, Q_X^{SQS}, Q_X^{GS}, Q_X^{SKS} and Q_X^{PVS} depend only upon the parameters of the Lévy processes $X_t^{(k)}$ and not in any way upon the time-change processes $Y_t^{(k)}$. It is well-known (Carr and Madan (1998)) that log-forward-contracts and entropy-forward-contracts can be replicated in a completely model-independent way by a continuum of co-terminal vanilla options of all strikes. Corollary 7.2 says that (provided Assumption (3.9) holds), given the prices of log-forward-contracts and entropy-forward-contracts, we can determine the prices of variance swaps, self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps in terms of the characteristic exponents of the Lévy processes $X_t^{(k)}$ and their derivatives. **We do not need any information whatsoever about the time-change processes $Y_t^{(k)}$. This gives considerable robustness to model (mis-)specification.**

Corollary 7.4. *In the special case of Theorem* 7.1 *when, for each $k = 1, 2, \ldots, K$, $X_t^{(k)}$ is a Brownian motion, then*

$$Q_X^{\mathrm{VS}} = 2, \quad Q_X^{\mathrm{SQS}} = 2, \quad Q_X^{\mathrm{GS}} = 2, \quad Q_X^{\mathrm{SKS}} = 0, \quad Q_X^{\mathrm{PVS}} = 2.$$

Proof: For each $k = 1, 2, \ldots, K$, $X_t^{(k)}$ is a Brownian motion with volatility σ, say, (which we can assume is the same for all k since the time-change processes $Y_t^{(k)}$ are general). We have, for each k, $\psi_k(z) = \sigma^2(z^2 + iz)/2$, $m_k(0) = -\sigma^2/2$, $\psi_k^{''}(0) = \sigma^2$, $\psi_k^{''}(-i) = \sigma^2$, $\psi_k^{'''}(0) = 0$ and $\psi_k(-2i) = -\sigma^2$. $\bullet$

In words, the first equation of Corollary 7.4 states that, if the underlying stock price has continuous sample paths, the price of a continuously monitored variance swap is minus twice the price of a log-forward-contract. This result is very well-known (Neuberger (1990), (1994), (1996), Dupire (1993), Derman et. al. (1999), Carr and Lee (2009)), but our extended characteristic function methodology gives us an alternative proof. The second equation of Corollary 7.4 is also well-known (see, for example, Neuberger (1996), Carr and Lee (2010a), Lee (2008)).

Corollary 7.5. *In the special case of Theorem* 7.1 *when $K = 1$ and $X_t^{(1)}$ is a compound Poisson process with a fixed jump amplitude a (and with no diffusion component), then as $a \to 0$,*

$$Q_X^{\mathrm{VS}} = \frac{a^2}{(\exp(a) - 1 - a)} \approx 2\left(1 - \frac{a}{3}\right), \quad Q_X^{\mathrm{SQS}} = \frac{a^2 \exp(a)}{(1 + a\exp(a) - \exp(a))} \approx 2\left(1 + \frac{a}{3}\right),$$
$$Q_X^{\mathrm{GS}} = \frac{a^2 \exp(a)}{(\exp(a) - 1 - a)} \approx 2\left(1 + \frac{2a}{3}\right), \quad Q_X^{\mathrm{SKS}} = \frac{a^3}{(\exp(a) - 1 - a)} \approx 2a\left(1 - \frac{a}{3}\right),$$
$$Q_X^{\mathrm{PVS}} = \frac{(\exp(a) - 1)^2}{(\exp(a) - 1 - a)} \approx 2\left(1 + \frac{2a}{3}\right).$$

Proof: We let the intensity rate, under $\mathbb{Q}$, of the compound Poisson process be λ. We have $\psi_1(z) = -\lambda(\exp(iza) - 1) + iz\lambda(\exp(a) - 1)$, $m_1(0) = -\lambda(\exp(a) - 1 - a)$, $m_1(1) = -\lambda(\exp(a) - 1 - a\exp(a))$, $\psi_1^{''}(z) = \lambda a^2 \exp(iza)$, $\psi_1^{'''}(0) = i\lambda a^3$ and $\psi_1(-2i) = -\lambda(\exp(a) - 1)^2$. •

Clearly, Corollary 7.5 gives us a simple way of relating the value of the multipliers to the skewness of the Lévy process(es), at least for small skews, in the spirit of extending results in Section 4 of Carr and Lee (2009). They show that negatively skewed (under $\mathbb{Q}$) Lévy process(es) (which seems more likely to be the case in practice for stock options markets and corresponds to the case of $a < 0$) imply $Q_X^{\mathrm{VS}} > 2$. Corollary 7.5 shows that in this case, $Q_X^{\mathrm{SQS}} < 2$, $Q_X^{\mathrm{GS}} < 2$ and $Q_X^{\mathrm{PVS}} < 2$ i.e. these latter multipliers have the opposite sensitivity to a and to the skewness of the Lévy process(es).

What the characteristic function methodology described here can do is three-fold: We can extend existing results that only apply when there are no jumps in the underlying to when the dynamics of the log of the underlying stock price are driven by (possibly, multiple) time-changed Lévy processes. We can price discretely monitored versions of variance swaps, self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps. We can also price continuously monitored versions of these swaps (by letting $N \to \infty$), and hence we have a way of determining the error in pricing discretely monitored versions of these swaps as if they were continuously monitored.

On a negative point, in contrast to the log-contract replication approach of Neuberger (1990) and Dupire (1993), neither Theorem 7.1, nor Corollaries 7.2, 7.4 or 7.5 address how to hedge variance swaps (or the other types of swaps considered here). There are interesting things to be said about hedging these swaps under time-changed Lévy process dynamics but they are not to be found in Theorem 7.1. We refer the reader to a companion paper, Crosby (2009), for results on hedging.

8. Numerical examples

In this section, we will give some numerical examples which illustrate our analysis. Throughout this section, we assume that Assumption (3.9) is in force i.e. we assume a common time-change. We consider a generalised CGMY (see Carr et al. (2002)) process (generalised in the sense that we sometimes allow diferent C_{Up}, C_{Down} and Y_{Up} and Y_{Down} parameters and we allow also for a diffusion component).

We considered six different combinations of parameters of the generalised CGMY process (labelled "params" in table 1) as follows: In each case, we have a diffusion component with volatility 0.1 (labelled "Vol" in table 1). Params 1 and 2 are parameter estimates based on calibrations to the market prices of vanilla options on the S & P500 stock index (for September 2000 and March 2000 respectively) and are quoted from Carr and Lee (2009). Params 3, 4, 5 and 6 are parameter sets which we made up to illustrate our results. Params 3 implies negatively skewed stock returns (since $M > G$). In params 4 we reverse the "up" and "down" parameters compared to params 3, so stock returns would be (for the puposes of illustration) positively skewed with these parameters. In params 5 and 6, we have symmetric Lévy measures (since $M = G$), with the difference being that, with params 6, we typically have much larger jumps (since the values of $M = G$ are one quarter those in params 5). For all six different combinations of parameters, we scaled the C_{Up} and C_{Down} parameters so that the (annualised) continuously monitored variance swap rate expressed as a volatility is always exactly 0.25. These parameter values were also used in Crosby (2009). We time-change this generalised CGMY process by an integrated Heston (1993) process, whose activity rate y_t is of the form

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dz_t, \quad y_{t_0} \equiv y_0, \text{ with } y_0 > 0,$$

with z_t standard Brownian motion and $\kappa > 0$, $\eta > 0$ and $\lambda \geq 0$ constants. We allow the Brownian motion z_t driving the activity rate y_t to have correlation ρ with the diffusion component of the generalised CGMY process. We discuss our choice of the value of ρ shortly. As in Crosby (2009), and based on calibrations to the market prices of vanilla options on the S & P500 stock index, we chose $\lambda = 1.3612$, $\kappa = 0.3881$, $y_0 = 1$ and $\eta = 1$.

In table 1, we report the values of Q_X^{VS}, Q_X^{SQS}, Q_X^{GS}, Q_X^{SKS} and Q_X^{PVS}, computed using Corollary 7.2, for the six different combinations of parameters of the generalised CGMY process. We also report (in the columns labelled "VS", "SQVS" and "PVS" respectively), in equivalent volatility terms, the rates on continuously monitored variance swaps (which are 0.25 by construction), self-quantoed variance swaps and proportional variance swaps.

We now consider variance swaps, proportional variance swaps and skewness swaps with maturity T for three values of T, namely $T = 0.0625$, $T = 0.125$ and $T = 0.5$. We consider three different values of ρ, namely $\rho = -0.99$, $\rho = 0$ and $\rho = 0.99$. We chose these three values simply for illustration as they will give us an opportunity to see how sensitive our analysis is to non-zero covariance between the underlying returns process (i.e. the $X_t^{(k)}$) and the stochastic time-change process (i.e. $\int_{t_0}^t y_s ds$) over a range which is close to the maximum possible range of ρ from -1 to 1. We consider two possible choices, labelled (a) and (b), for the values of the interest-rate $r(t)$ and the dividend yield $q(t)$. In the first choice, labelled (a), we set, for simplicity, $r(s) = 0$, $q(s) = 0$, for all $s \in [t_0, T]$. In the second choice, labelled (b), we set $r(s) = 0.065$, $q(s) = 0.015$, for all $s \in [t_0, T]$. For the sake of brevity, we only report results for one of the six different combinations of parameters of the generalised CGMY process (params 2 in table 1).

We compute the prices of the three different types of swaps when the monitoring times are equally spaced between time $t_0 \equiv 0$ and maturity T and when the number N of monitoring times has values 2^{J-1} for $J = 1, \ldots, 9$. The results, for params 2, are reported graphically in figures 2, 3 and 4 for the eight different combinations generated as follows: For $T = 0.5$, we consider the two choices (a) and (b) and the three possible choices for ρ. This gives us a total of six of the eight different combinations. For the seventh and eighth different combinations, we vary the maturity T. We set $T = 0.125$ and $T = 0.0625$ respectively and use only choice (a) and $\rho = -0.99$ for both maturities. The results are reported in equivalent volatility terms (expressed as a percentage) for variance swaps (i.e. as $\sqrt{(\text{VS}(t_0, T, N)/P(t_0, T))/(T - t_0)}$ in figure 2) and proportional variance swaps (i.e. as $\sqrt{(\text{PVS}(t_0, T, N)/P(t_0, T))/(T - t_0)}$ in figure 3) and as undiscounted prices for skewness swaps (i.e. as $\text{SKS}(t_0, T, N)/P(t_0, T)$ in figure 4, for $N = 2^{J-1}$).

We now discuss the numerical results in table 1 and figures 2, 3 and 4:

We see that, essentially, the values of Q_X^{VS}, Q_X^{SQS}, Q_X^{GS} and Q_X^{PVS} are approximately equal to two, more than two or less than two according to the skewness of the Lévy process as suggested by the approximate terms in Corollary 7.5. As an example, the values of Q_X^{PVS} are very much less than two (approximately 1.6 to 1.8) for params 1 and 2 which, we repeat, are based on calibrations to market data, and imply significant negative skewness in the Lévy process. We conclude that jumps, especially asymmetric jumps, significantly affect the prices of the different types of swaps considered here.

When considering the impact of discrete monitoring on the prices of variance swaps, skewness swaps and proportional variance swaps, it is important to relate the size of the impact to a possible bid-offer spread that an investment bank might charge if asked to make a two-way price in the relevant swap. Based on conversations with traders, we expect the bid-offer spread for variance swaps to be approximately 0.2

percentage points in volatility terms, the bid-offer spread for proportional variance swaps to be approximately 0.4 percentage points in volatility terms and the bid-offer spread for skewness swaps to be approximately 3 per cent in proportional price terms for swaps quoted on major stock indices (or on futures contracts on major stock indices).

Note that daily monitoring (based on working days) approximately equates to $J = 8$ (corresponding to $N = 2^7 = 128$ monitoring times) for $T = 0.5$, to $J = 6$ for $T = 0.125$ and to $J = 5$ for $T = 0.0625$. From figure 2, we see that for daily monitoring (which is the most common situation in practice), the difference in price between a continuously monitored variance swap and its discretely monitored counterpart is typically less than 0.03 percentage points in volatility terms or approximately one seventh of the bid-offer spread. With slightly more than weekly monitoring (for $T = 0.5$, this is approximated by $J = 6$ (corresponding to $2^5 = 32$ monitoring times) and for $T = 0.125$, this is approximated by $J = 4$) - swaps with weekly monitoring do occasionally trade in the markets but they are less common, then discrete monitoring has much more impact (up to approximately 0.11 percentage points in volatility terms i.e. more than half the bid-offer spread) on the prices of variance swaps. With slightly more than monthly monitoring (for $T = 0.5$, this is approximated by $J = 4$ (corresponding to $2^3 = 8$ monitoring times)), then the impact of discrete monitoring is greater still - sometimes more than twice the bid-offer spread.

With daily monitoring, there is a greater impact on the prices of proportional variance swaps (around 0.06 percentage points in volatility terms in some cases), and skewness swaps (around 1.2 per cent in proportional price terms). In the case of skewness swaps, this is close to half the bid-offer spread, which suggests against a naive assumption that the impact of discrete monitoring, for the daily case, will always be negligible. Changing the correlation ρ has more impact than changing the risk-neutral drift.

Overall, however, it is clear that the impact of jumps is much greater than that of discrete monitoring. This has already been remarked upon by Broadie and Jain (2008a) for variance swaps. But, it is even clearer for proportional variance swaps and skewness swaps. Recall that we scaled the generalised CGMY parameters so that the (annualised) continuously monitored variance swap rate expressed as a volatility is always exactly 0.25. In the absence of jumps (when the Lévy processes are all Brownian motions), the (annualised) continuously monitored proportional variance swap rate expressed as a volatility would also be exactly 0.25 (from Corollary 7.4) - in contrast to the actual rate of approximately 0.207546 with our generalised CGMY parameters (which we stress again are based on calibrations to the market prices of vanilla options on the S & P500 stock index). In the absence of jumps, the price of the continuously monitored skewness swap would be identically zero.

Finally, we remark that detailed analysis (not reported) confirms the $O(\Delta_N) = O(1/N)$ convergence rates seen in Propositions 6.2, 6.5 and 6.6.

9. Results if the stochastic time-change processes are discontinuous

Until now, we have assumed that the time-change processes $Y_t^{(k)}$ satisfy Assumptions (3.4) and (6.1). In particular, Assumption (6.1) required that the time-change processes $Y_t^{(k)}$ were continuous processes.

In this section **(and not in any other section)**, we consider what would happen if the time-change processes $Y_t^{(k)}$ are allowed to be stochastic and discontinuous. In this section only, we, therefore, extend Assumption (6.1) as follows to allow the possibility that, for a given k, $Y_t^{(k)}$ may be discontinuous (eg. a gamma process). We continue to assume Assumption (3.4) throughout.

Assumption 9.1. We assume that, for each $k = 1, 2, \ldots, K$, either $Y_t^{(k)}$ satisfies Assumption (6.1) or $Y_t^{(k)}$ satisfies the following extension: We allow $Y_t^{(k)}$ to be a pure-jump non-decreasing process, with Lévy measure denoted by $\nu_Y^{(k)}(du, dt)$, which has independent (but not necessarily stationary) increments, i.e. $Y_t^{(k)}$ is a (possibly time-inhomogeneous) additive process and a subordinator implying that it has finite variation. If, for a given k, $Y_t^{(k)}$ satisfies Assumption (6.1), we write $\mathbf{1}_k(d) = 0$ (the "d" is for discontinuous), whereas if $Y_t^{(k)}$ is discontinuous as just specified, we write $\mathbf{1}_k(d) = 1$. If $\mathbf{1}_k(d) = 1$, for some k, we asssume that $Y_t^{(k)}$ is independent of $Y_t^{(j)}$, for all $k \neq j$, and that $Y_t^{(k)}$ is independent of $X_t^{(\ell)}$, for all $\ell = 1, 2, \ldots, K$.

Note that the Lévy measure $\nu_Y^{(k)}(du, dt)$ may be time-dependent. Hence, Lévy measure may be, to some, a misnomer. However, we will use this term (following Cont and Tankov (2004), see their Theorem 14.1).

We only consider variance swaps, partly for brevity but mainly because we do not have anything like a full set of answers. Under the extended Assumption (9.1), one can see that equation (5.1) is still valid for the price of a discretely monitored variance swap. However, the value to which this price converges in the limit that $N \to \infty$ is not necessarily the same as before. We state the result in the next proposition.

Proposition 9.2. *Under the extended Assumption* (9.1)*, then*

$$\begin{aligned}\lim_{N\to\infty} \mathrm{VS}(t_0, T, N) &= P(t_0, T) \sum_{k=1}^{K} \psi_k^{''}(0) \mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T^{(k)} - Y_{t_0}^{(k)}] \\ &+ P(t_0, T) \sum_{k=1}^{K} \mathbf{1}_k(d)(m_k(0))^2 \int_{t_0}^{T} \int_0^{\infty} u^2 \nu_Y^{(k)}(du, ds), \end{aligned} \tag{9.1}$$

and

$$|\mathrm{VS}(t_0, T, N) - \lim_{N\to\infty} \mathrm{VS}(t_0, T, N)| = O(\Delta_N) \text{ as } N \to \infty. \tag{9.2}$$

Proof: See the Appendix. •

The first term in equation (9.1) is as in equation (6.5). Bur we now have a second term present when there are discontinuous stochastic time-change process(es) satisfying Assumption (9.1). Note however that the convergence rate in equation (9.2) remains $O(\Delta_N)$. In other words, when we have discontinuous stochastic time-change process(es), the value to which the price of a discretely monitored variance swap converges as $N \to \infty$ changes compared to the case of continuous stochastic time-change process(es) (i.e. satisfying Assumption (6.1)) but the convergence rate is the same.

A very wide class of semimartingale processes can be expressed as Lévy processes time-changed by processes satisfying Assumption (9.1). This leads us to conjecture that the $O(\Delta_N)$ convergence rate for variance swap prices applies in even greater generality - possibly when the log of the stock price is any semimartingale (for which the relevant expectations are well-defined). We thank a referee for drawing our attention to a recent paper (Jarrow et al. (2011)) which goes some way to proving our conjecture.

10. Conclusions

We have examined the pricing of variance swaps and some generalisations and variants such as self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps. We have considered the convergence of the prices of discretely monitored versions of these swaps to the prices of their continuously monitored counterparts as the number N of monitoring times tends to infinity. We have significantly extended results in Broadie and Jain (2008a) by showing that, under very general stock price dynamics, the prices of discretely monitored variance swaps and all of their generalisations and variants listed above all converge to those of their continuously monitored counterparts as $O(1/N)$. We have generalised results in Carr and Lee (2009) relating the prices of variance swaps and the generalisations and variants listed above to those of log-forward-contracts and entropy-forward-contracts.

Appendix A. Proof of convergence results

A.1. **Convergence results. We stress that we assume that the stochastic time-change processes $Y_t^{(k)}$ are continuous as in Assumption (6.1) throughout the appendix except in the proof of Proposition 9.2.**

We first state a preliminary lemma.

Lemma A.1.

$$\sum_{j=1}^{n}\mathbb{E}^{\mathbb{Q}}_{t_0}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)^2] \ \textit{is}\ O(\Delta_n), \qquad \sum_{j=1}^{n}\big(\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}]\big)^2 \ \textit{is}\ O(\Delta_n),\ \textit{and} \tag{A.1}$$

$$\mathbb{V}ar^{\mathbb{Q}}_{t_{j-1,(n)}}[\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})] \ \textit{is}\ O(\Delta_n^2). \tag{A.2}$$

$$\begin{aligned}\Big|\sum_{j=1}^{n}\mathbb{E}^{\mathbb{Q}}_{t_0}[\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}(\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i))-1)]-\\ -\sum_{k=1}^{K}\psi_k(-2i)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T-Y^{(k)}_{t_0}]\Big| \ \textit{is}\ O(\Delta_n).\end{aligned} \tag{A.3}$$

Proof: Equations $(A.1)$ and $(A.2)$ are immediate from equation (6.3) and the assumption that $\Delta_n = O(1/n)$. Equation $(A.3)$ is immediate from equation (6.17). •

Proof of Proposition 6.2: A variance swap corresponds to the case (see equation (4.6)) that Υ_1, Υ_2 and Υ_3 are all zero. We consider the convergence of $\Omega_1(0,0,0,n)$, $\Omega_2(0,0,0,n)$, $\Omega_3(0,0,0,n)$ and $\Omega_4(0,0,0,n)$ in turn: We have the following results:

For the term $\Omega_1(0,0,0,n)$ (which we remark is certainly non-negative):

$$\begin{aligned}
\Omega_1(0,0,0,n) &= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}}_{t_0}[\varpi^{(j)\,2}(0)] \\
&= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}}_{t_0}[\big(\int_{t_{j-1,(n)}}^{t_{j,(n)}} (r(s)-q(s))ds + \sum_{k=1}^{K} m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\big)^2] \\
&= \sum_{j=1}^{n} \big(\int_{t_{j-1,(n)}}^{t_{j,(n)}} (r(s)-q(s))ds\big)^2 \\
&+ 2\sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{k=1}^{K} m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})] \int_{t_{j-1,(n)}}^{t_{j,(n)}} (r(s)-q(s))ds \\
&+ \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}}_{t_0}[\big(\sum_{k=1}^{K} m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\big)^2].
\end{aligned} \tag{A.4}$$

Using equations (6.3) and $(A.1)$, it is clear that $\Omega_1(0,0,0,n)$ is $O(\Delta_n)$.

We considered the term $\Omega_2(0,0,0,n)$ in Remark 5.2 and more specifically in equation (5.2). For any two random variables Z_1 and Z_2, we have $\text{Covar}[Z_1,Z_2] \le (\mathbb{V}ar[Z_1]\mathbb{V}ar[Z_2])^{1/2}$. Now we apply this result to $\Omega_2(0,0,0,n)$ as re-expressed in equation (5.2)

$$\begin{aligned}
|\Omega_2(0,0,0,n)| \le \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[(\mathbb{V}ar^{\mathbb{Q}}_{t_{j-1,(n)}}[2\sum_{k=1}^{K} m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})] \times \\
\mathbb{V}ar^{\mathbb{Q}}_{t_{j-1,(n)}}[\mathbb{E}^{\mathbb{Q}}[\sum_{k=1}^{K}\Big((X^{(k)}_{Y_{t_{j,(n)}}} - X^{(k)}_{Y_{t_{j-1,(n)}}}) - m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\Big) \,| \\
\mathcal{F}_{t_{j-1}} \vee \sigma\{Y^{(k)}_{t_j}, k=1,2,\ldots,K\}]])^{1/2}\Big].
\end{aligned}$$

Since $\mathbb{V}ar^{\mathbb{Q}}_{t_{j-1,(n)}}[2\sum_{k=1}^{K} m_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})]$ is (from equation $(A.2)$) $O(\Delta_n^2)$ and using equation (6.4), it is clear that $|\Omega_2(0,0,0,n)|$ is $O(\Delta_n)$.

We have already noted that the term $\Omega_3(0,0,0,n)$ is identically zero (by equation (3.10)).

Finally, we consider the term $\Omega_4(0,0,0,n)$:

$$\Omega_4(0,0,0,n) = \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}}_{t_0}\Big[\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\sum_{k=1}^{K} \psi''_k(0)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})]\Big] = \sum_{k=1}^{K} \psi''_k(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}].$$

Clearly, $\Omega_4(0,0,0,n)$ is strictly positive and, furthermore, it is independent of n.

Adding together the four terms gives the result. •

Proof of Proposition 6.3**:** We focus initially on self-quantoed variance swaps. A self-quantoed variance swap corresponds to the case that Υ_1, Υ_2 and Υ_3 are all one. We consider the convergence of $\Omega_1(1,1,1,n)$, $\Omega_2(1,1,1,n)$, $\Omega_3(1,1,1,n)$ and $\Omega_4(1,1,1,n)$ in turn. Then we have the following results.

To compute the limit as $n \to \infty$ of $\Omega_1(1,1,1,n)$ and $\Omega_2(1,1,1,n)$, we change probability measure to $\mathbb{Q}\,(1)$. Then, the rest of the proof goes through essentially as in Proposition 6.2. Again, $\Omega_3(1,1,1,n) = 0$.

Furthermore, using the fact that $\psi_k(-i) = 0$, and with some algebraic rearrangement,

$$\begin{aligned}\Omega_4(1,1,1,n) &= D(t_0,T,1)\sum_{j=1}^{n}\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(\sum_{k=1}^{K}(X^{(k)}_{Y_T} - X^{(k)}_{Y_{t_0}}))\sum_{k=1}^{K}\psi''_k(-i)(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})] \\ &= \sum_{k=1}^{K}\psi''_k(-i)\mathbb{E}^{\mathbb{Q}}_{t_0}[\frac{F(T,T)}{F(t_0,T)}(Y^{(k)}_T - Y^{(k)}_{t_0})].\end{aligned}$$

Clearly, $\Omega_4(1,1,1,n)$ is strictly positive and, furthermore, it is independent of n.

Adding together the four terms, the parts of the proposition relating to self-quantoed variance swaps are proven. The proof of the parts of the proposition relating to gamma swaps is very similar and so is omitted. •

Proof of Proposition 6.5**:** We differentiate equation (4.5) with respect to z_2 and again divide by i. We set $iz_1 = 0$, $iz_2 = 0$ and $iz_3 = 0$, rearrange and sum over j. This gives an explicit formula (which is rather long, so omitted for brevity) for the price of a discretely monitored skewness swap. As we let the number of monitoring times tend to infinity, one sees that all but one term vanishes. This term is the right-hand-side of equation (6.14), proving the first part of the proposition. The second part of the proposition follows as in the proofs of Propositions 6.2 and 6.3 (using, in particular, equations (6.12) and (6.13) - again, we omit the details for brevity). •

Proof of Proposition 6.6**:** To start with we assume that $r(t) = q(t)$. Hence in this special case,

$$\mathrm{PVS}(t_0,T,n) = P(t_0,T)\Big(\sum_{j=1}^{n}\big(\Phi(0,-2i,0;j) - 1\big)\Big).$$

But from the form of $\Phi(0,-2i,0;j)$ (see equation (3.8)), we have

$$\begin{aligned}\mathrm{PVS}(t_0,T,n) &= P(t_0,T)\Big(\sum_{j=1}^{n}\mathbb{E}^{\mathbb{Q}}_{t_0}[\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}(\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i)) - 1)]]\Big) \\ &= P(t_0,T)\Big(\sum_{j=1}^{n}\mathbb{E}^{\mathbb{Q}}_{t_0}[\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}(\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i)) - 1)]\Big).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{PVS}(t_0,T,\infty) &= P(t_0,T)\lim_{n\to\infty}\Big(\sum_{j=1}^{n}\mathbb{E}^{\mathbb{Q}}_{t_0}[\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}(\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}} - Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i)) - 1)]\Big) \\ &= -P(t_0,T)\sum_{k=1}^{K}\psi_k(-2i)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y^{(k)}_T - Y^{(k)}_{t_0}],\end{aligned}$$

by equation $(A.3)$. This proves equation (6.18). Furthermore,

$$|\mathrm{PVS}(t_0,T,n) - \mathrm{PVS}(t_0,T,\infty)| \text{ is } O(\Delta_n) \text{ as } n \to \infty,$$

giving equation (6.19).

We recall that we assume that $r(t)$ and $q(t)$ are deterministic and finite, for all $t \geq t_0$. Hence, extending the results above to the case when $r(t) \neq q(t)$ is straightforward by Taylor expansion. •

We will now prove Proposition 6.8. It will shorten the proof if we first state three auxiliary lemmas, which use special cases of the dynamics for $Y^{(k)}_t$ assumed in Proposition 6.8.

Lemma A.2. *Suppose that in the statement of Proposition* 6.8*, we have the special cases* $\alpha_{mk}=0$ *and* $\beta_{mk}=0$ *for all* m *and* k*, i.e.* $Y_t^{(k)}$*, for each* k*, is actually deterministic. Then*

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)] \text{ is } O(\Delta_n),$$
$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)\big(Y^{(\ell)}_{t_{j,(n)}}-Y^{(\ell)}_{t_{j-1,(n)}}\big)] \text{ is } O(\Delta_n^2), \tag{A.5}$$

and, furthermore, equations (6.12)*,* (6.13) *and* (6.17) *hold.*

Proof: Immediate - the proof of equation (6.13) follows using equation (3.10) and that of equation (6.17) follows by Taylor expansion. •

Lemma A.3. *Suppose that in the statement of Proposition* 6.8*, we have the special cases* $g_k(t)=0$*,* $\alpha_{mk}=0$*,* $M=K$*,* $\beta_{mk}=\delta_{mk}$ *. This diagonalises the form of* $Y_t^{(k)}$*, so that each* $Y_t^{(k)}$ *follows an independent integrated Heston (1993) process. Then*

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)] \text{ is } O(\Delta_n),$$
$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)\big(Y^{(\ell)}_{t_{j,(n)}}-Y^{(\ell)}_{t_{j-1,(n)}}\big)] \text{ is } O(\Delta_n^2), \tag{A.6}$$

and, furthermore, equation (6.17) *holds.*

Proof: We can (see, for example, Broadie and Jain (2008a) or Itkin and Carr (2010), their equation (26)) compute an explicit expression for $\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)]$:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)] &= \eta^{(k)}(t_{j,(n)}-t_{j-1,(n)})+\frac{(y^{(k)}_{t_{j-1,(n)}}-\eta^{(k)})}{\kappa^{(k)}}(1-\exp(-\kappa^{(k)}(t_{j,(n)}-t_{j-1,(n)})))\\ &= y^{(k)}_{t_{j-1,(n)}}(t_{j,(n)}-t_{j-1,(n)})+\ldots\\ &= O(\Delta_n),\end{aligned}$$

where the second line follows by Taylor expansion.

An explicit expression for $\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)^2]$ is computed in Broadie and Jain (2008a) (see their equation (A.22)) and shown to be $O(\Delta_n^2)$ (see between their equations (A.22) and (A.24)) by another Taylor expansion. This implies $\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)\big(Y^{(\ell)}_{t_{j,(n)}}-Y^{(\ell)}_{t_{j-1,(n)}}\big)]$ is $O(\Delta_n^2)$ for all k and ℓ since, in the special case considered in this lemma, $Y_t^{(k)}$ is independent of $Y_t^{(\ell)}$ for $\ell\neq k$.

An explicit form for the Laplace transform of $(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})$ is given in Cont and Tankov (2004) (see Section 15.1). Because, $Y_t^{(k)}$ is independent of $Y_t^{(\ell)}$ for $k\neq\ell$, we get the Laplace transform of $\sum_{k=1}^K(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})$ as a product of the Laplace transforms of $(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})$, for each k. We expand the Laplace transform by Taylor expansion. Then

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[(\exp(-\sum_{k=1}^K(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i))-1)]\\ &=-(t_{j,(n)}-t_{j-1,(n)})\sum_{k=1}^K\psi_k(-2i)y^{(k)}_{t_{j-1,(n)}}+D_0(t_{j,(n)}-t_{j-1,(n)})^2+\ldots,\end{aligned} \tag{A.7}$$

where the remainder is of third order and D_0 depends on the parameters of the $y_t^{(k)}$ processes. We could use the martingale $\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}$ to change measure. Then, Carr and Wu (2004), Lee (2009) and Andersen and

Piterbarg (2007)) show that $y_t^{(k)}$ remains a Heston (1993) process under the new measure with, in general, different parameters. So

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\frac{\Xi_{t_{j,(n)}}(-2i)}{\Xi_{t_{j-1,(n)}}(-2i)}(\exp(-\sum_{k=1}^{K}(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}})\psi_k(-2i))-1)] \\
&= -(t_{j,(n)}-t_{j-1,(n)})\sum_{k=1}^{K}\psi_k(-2i)y^{(k)}_{t_{j-1,(n)}}+D_2(t_{j,(n)}-t_{j-1,(n)})^2+\ldots, \qquad \text{(A.8)}
\end{aligned}$$

The first term in the second line of equation $(A.8)$ is the same as the first term in the second line of equation $(A.7)$ since $y^{(k)}_{t_{j-1,(n)}}$ is invariant to the change of measure. Rearranging equation $(A.8)$, using equation $(A.7)$, and taking $\mathbb{Q}$ expectations at time t_0 gives equation (6.17). •

Lemma A.4. *If in Proposition* 6.8*, we have the special cases* $g_k(t)=0$*,* $\beta_{mk}=0$*,* $M=K$*,* $\alpha_{mk}=\delta_{mk}$ *(this diagonalises the form of* $Y^{(k)}_t$ *so that each* $Y^{(k)}_t$ *follows an independent integrated non-Gaussian OU process), then*

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)] &\ \textit{is}\ O(\Delta_n), \\
\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)\big(Y^{(\ell)}_{t_{j,(n)}}-Y^{(\ell)}_{t_{j-1,(n)}}\big)] &\ \textit{is}\ O(\Delta_n^2), \qquad \text{(A.9)}
\end{aligned}$$

and, furthermore, equation (6.17) *holds.*

Proof: We can (see, for example, Proposition 15.2 of Cont and Tankov (2004) or the beginning of Section 8.3 of Schoutens (2003)) compute an explicit expression for $y^{(k)}_u$ and for $\mathbb{E}^{\mathbb{Q}}_t[y^{(k)}_u]$, for $u \geq t$:

$$\begin{aligned}
y^{(k)}_u &= y^{(k)}_t\exp(-\theta^{(k)}(u-t))+\int_t^u \exp(\theta^{(k)}(s-u)))dN^{(k)}_s, \\
\mathbb{E}^{\mathbb{Q}}_t[y^{(k)}_u] &= y^{(k)}_t\exp(-\theta^{(k)}(u-t))+\frac{\mathbb{E}^{\mathbb{Q}}[N^{(k)}_1]}{\theta^{(k)}}(1-\exp(-\theta^{(k)}(u-t))),
\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}}[N^{(k)}_1]$ is the mean of $N^{(k)}_1$ (ie the $\mathbb{Q}$ expectation of the time 1 value of $N^{(k)}_t$ given its time 0 value). Hence:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\big(Y^{(k)}_{t_{j,(n)}}-Y^{(k)}_{t_{j-1,(n)}}\big)] &= \mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[\int_{t_{j-1,(n)}}^{t_{j,(n)}} y^{(k)}_u du] \\
&= \int_{t_{j-1,(n)}}^{t_{j,(n)}} \mathbb{E}^{\mathbb{Q}}_{t_{j-1,(n)}}[y^{(k)}_u]du \\
&= \frac{y^{(k)}_{t_{j-1,(n)}}}{\theta^{(k)}}(1-\exp(-\theta^{(k)}(t_{j,(n)}-t_{j-1,(n)}))) \\
&- \frac{\mathbb{E}^{\mathbb{Q}}[N^{(k)}_1]}{\theta^{(k)\,2}}(1-\exp(-\theta^{(k)}(t_{j,(n)}-t_{j-1,(n)}))) \\
&+ \frac{\mathbb{E}^{\mathbb{Q}}[N^{(k)}_1]}{\theta^{(k)}}(t_{j,(n)}-t_{j-1,(n)}) \\
&= y^{(k)}_{t_{j-1,(n)}}(t_{j,(n)}-t_{j-1,(n)})+\ldots \\
&= O(\Delta_n),
\end{aligned}$$

where we have directly integrated and then used Taylor expansion. Likewise, we can compute an explicit expression for $\mathbb{E}_t^{\mathbb{Q}}[y_u^{(k)} y_v^{(k)}]$, for $u, v \geq t$:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}[y_u^{(k)} y_v^{(k)}] &= y_t^{(k)\,2} \exp(-\theta^{(k)}(u+v-2t)) \\
&+ y_t^{(k)} \frac{\mathbb{E}^{\mathbb{Q}}[N_1^{(k)}]}{\theta^{(k)}} (1-\exp(-\theta^{(k)}(u-t))) \exp(-\theta^{(k)}(v-t)) \\
&+ y_t^{(k)} \frac{\mathbb{E}^{\mathbb{Q}}[N_1^{(k)}]}{\theta^{(k)}} (1-\exp(-\theta^{(k)}(v-t))) \exp(-\theta^{(k)}(u-t)) \\
&+ \frac{\mathbb{E}^{\mathbb{Q}}[N_1^{(k)\,2}]}{\theta^{(k)\,2}} (1-\exp(-\theta^{(k)}(\min(u,v)-t)))^2 \\
&= y_t^{(k)\,2} + \ldots,
\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}}[N_1^{(k)\,2}]$ is the mean of $N_1^{(k)\,2}$. Now

$$\begin{aligned}
\mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[\big(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)}\big)^2] &= \mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[\Big(\int_{t_{j-1,(n)}}^{t_{j,(n)}} y_u^{(k)} du\Big)\Big(\int_{t_{j-1,(n)}}^{t_{j,(n)}} y_v^{(k)} dv\Big)] \\
&= \mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[\int_{t_{j-1,(n)}}^{t_{j,(n)}} \int_{t_{j-1,(n)}}^{t_{j,(n)}} y_u^{(k)} y_v^{(k)} du dv] \\
&= \int_{t_{j-1,(n)}}^{t_{j,(n)}} \int_{t_{j-1,(n)}}^{t_{j,(n)}} \mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[y_u^{(k)} y_v^{(k)}] du dv \\
&= \int_{t_{j-1,(n)}}^{t_{j,(n)}} \int_{t_{j-1,(n)}}^{t_{j,(n)}} (y_{t_{j-1,(n)}}^{(k)\,2} + \ldots) du dv \\
&= y_{t_{j-1,(n)}}^{(k)\,2} (t_{j,(n)} - t_{j-1,(n)})^2 + \ldots \\
&= O(\Delta_n^2).
\end{aligned}$$

This further implies $\mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[\big(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)}\big)\big(Y_{t_{j,(n)}}^{(\ell)} - Y_{t_{j-1,(n)}}^{(\ell)}\big)]$ is $O(\Delta_n^2)$ for all k and ℓ since, in the special case considered in this lemma, $Y_t^{(k)}$ is independent of $Y_t^{(\ell)}$ for $\ell \neq k$.

The proof that equation (6.17) holds is identical to that in Lemma $A.3$ (the Laplace transform of $(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})$ is given in Cont and Tankov (2004), see Section 15.3.3; furthermore, the change of measure argument also works (as in Carr and Wu (2004)), Details are omitted. •

Proof of Proposition 6.8**:**

To begin with, we consider the case where $\mathbb{Q}(\varphi)$ is $\mathbb{Q}$.

Proof when $\mathbb{Q}(\varphi)$ is $\mathbb{Q}$

The first part of equation (6.3) holds from Lemmas $A.2$, $A.3$ and $A.4$ and linearity of expectation.

The second part of equation (6.3) holds from Lemmas $A.2$, $A.3$ and $A.4$ and the covariance structure in equation (6.20).

Now we consider equation (6.4) (we continue to assume, for the moment, that $\mathbb{Q}(\varphi)$ is $\mathbb{Q}$). From Proposition 3.13 of Cont and Tankov (2004),

$$\mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[\sum_{k=1}^{K} \Big((X_{Y_{t_{j,(n)}}}^{(k)} - X_{Y_{t_{j-1,(n)}}}^{(k)}) - m_k(0)(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)}) \Big) \mid$$
$$\mathcal{F}_{t_{j-1}} \vee \sigma\{Y_{t_j}^{(k)}, k=1,2,\ldots,K\}] = \sum_{k=1}^{K} C_k (Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)}),$$

where, for each k, C_k depends only on the k^{th} Lévy triplet and, in particular, is independent of $Y_t^{(k)}$. Equation (6.4) then follows using both parts of equation (6.3).

Hence, we have shown that equations (6.3) and (6.4) hold for the special case that $\mathbb{Q}(\varphi)$ is $\mathbb{Q}$.

Proof when $\mathbb{Q}(\varphi)$ is $\mathbb{Q}(1)$

We now extend the proof to the case when $\mathbb{Q}(\varphi)$ is $\mathbb{Q}(1)$. But this follows because Carr and Wu (2004) (see also Lee (2009) and Andersen and Piterbarg (2007)) show that if the activity rate $y_t^{(k)}$ is an affine jump-diffusion process under $\mathbb{Q}$, then it remains an affine jump-diffusion process under $\mathbb{Q}(1)$ (with, in general, different parameters - but this does not alter the convergence rate). This completes the proof of equations (6.3) and (6.4).

Equation (6.17) holds from the arguments in Lemmas $A.2$, $A.3$ and $A.4$ and the covariance structure in equation (6.20). •

All the convergence results above (i.e. the proofs of Propositions 6.2, 6.3, 6.5 and 6.6) assume that the stochastic time-change processes $Y_t^{(k)}$ are continuous as in Assumption (6.1). We now consider the convergence of variance swap prices if the stochastic time-change processes $Y_t^{(k)}$ are allowed to be discontinuous as in Assumption (9.1). Specifically, we provide the proof of Proposition 9.2.

Proof of Proposition 9.2**:** The terms $\Omega_2(0,0,0,n)$, $\Omega_3(0,0,0,n)$ and $\Omega_4(0,0,0,n)$ are all as in Proposition 6.2 and converge as in the proof of that proposition. So the only term we need to consider in detail is $\Omega_1(0,0,0,n)$. The form of $\Omega_1(0,0,0,n)$ is exactly as in equation $(A.4)$. As $n \to \infty$, the first line in equation $(A.4)$ tends to zero as $O(\Delta_n)$ (exactly as in the proof of Proposition 6.2). If $\mathbf{1}_k(d) = 0$, then $\sum_{j=1}^{n} \mathbb{E}_{t_0}^{\mathbb{Q}}[m_k(0)(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})] \int_{t_{j-1,(n)}}^{t_{j,(n)}} (r(s)-q(s))ds$ tends to zero as $O(\Delta_n)$ (because of the first part of equation (6.3)). If, on the other hand, $\mathbf{1}_k(d) = 1$, then $\sum_{j=1}^{n} \mathbb{E}_{t_0}^{\mathbb{Q}}[m_k(0)(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})] \int_{t_{j-1,(n)}}^{t_{j,(n)}} (r(s) - q(s))ds$ also tends to zero as $O(\Delta_n)$ since in this case $\mathbb{E}_{t_{j-1,(n)}}^{\mathbb{Q}}[(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})]$ is also $O(\Delta_n)$ (from results in Section 3.6 of Cont and Tankov (2004)). So, in all cases, the second line in equation $(A.4)$ tends to zero as $O(\Delta_n)$.

For the final line of equation $(A.4)$, we have that, as $n \to \infty$:

If $\mathbf{1}_k(d) = 0$, then $\sum_{j=1}^{n} \mathbb{E}_{t_0}^{\mathbb{Q}}[\big(\sum_{k=1}^{K} m_k(0)(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})\big)^2]$ tends to zero as $O(\Delta_n)$ (exactly as in the proof of Proposition 6.2).

On the other hand, if $\mathbf{1}_k(d) = 1$, then observing that $\lim_{n\to\infty} \sum_{j=1}^{n} (Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})^2$ is the quadratic variation of $Y_t^{(k)}$, we use a a well-known result (p7 of Bertoin (1996)) for its expectation, to conclude that:

$$\lim_{n\to\infty} P(t_0,T) \sum_{j=1}^{n} \mathbb{E}_{t_0}^{\mathbb{Q}}[\big(\sum_{k=1}^{K} m_k(0)(Y_{t_{j,(n)}}^{(k)} - Y_{t_{j-1,(n)}}^{(k)})\big)^2] =$$

$$P(t_0,T) \sum_{k=1}^{K} \mathbf{1}_k(d)(m_k(0))^2 \int_{t_0}^{T} \int_0^{\infty} u^2 \nu_Y^{(k)}(du, ds). \tag{A.10}$$

Furthermore, the convergence rate is $O(\Delta_n)$ (from results in Section 3.6 of Cont and Tankov (2004)). Putting everything together, this concludes the proof. •

Below is table 1 which presents numerical results discussed in Section 8.

Table 1.

	C_{Up}	C_{Down}	G	M	Y_{Up}	Y_{Down}	Vol
params 1	0.50637884	0.03423121	1.64	16.9	-2.9	1.54	0.1
params 2	0.09238822	0.02663552	0.697	22	-3.65	1.45	0.1
params 3	0.07465977	0.50505138	3.34	14.64	0.165	0.165	0.1
params 4	0.50505138	0.07465977	14.64	3.34	0.165	0.165	0.1
params 5	9.10368153	9.10368153	22.56	22.56	0.14	0.14	0.1
params 6	0.69085272	0.69085272	5.64	5.64	0.14	0.14	0.1

	Q_X^{VS}	Q_X^{SQS}	Q_X^{GS}	Q_X^{SKS}	Q_X^{PVS}	VS	SQVS	PVS
params 1	2.1388910	1.8958074	1.7857160	-0.5037470	1.7978118	0.25	0.2284293	0.2292017
params 2	2.3469497	1.8015140	1.5839867	-1.5556476	1.6175307	0.25	0.2053827	0.2075460
params 3	2.2873888	1.7753472	1.5639307	-1.0430172	1.5919865	0.25	0.2067182	0.2085642
params 4	1.6748270	2.4459122	2.9566032	0.7636976	3.1960757	0.25	0.3321632	0.3453533
params 5	1.9985360	2.0043982	2.0073363	0	2.0088275	0.25	0.2505498	0.2506429
params 6	1.9763984	2.0724377	2.1223373	0	2.1538973	0.25	0.2590657	0.2609848

Figure 1 shows the "empirical" value of the multiplier Q_X^{VS} (i.e. the value implied by dividing the market prices of variance swaps by (minus) the prices of log-forward-contracts inferred from the market prices of vanilla options) based on market transactions recorded on 10th December 2010 for variance swaps written on the Nikkei-225 stock index with maturities of one month, two months, three months, four months and six months.

Figure 1. "Empirical" values of Q_X^{VS}

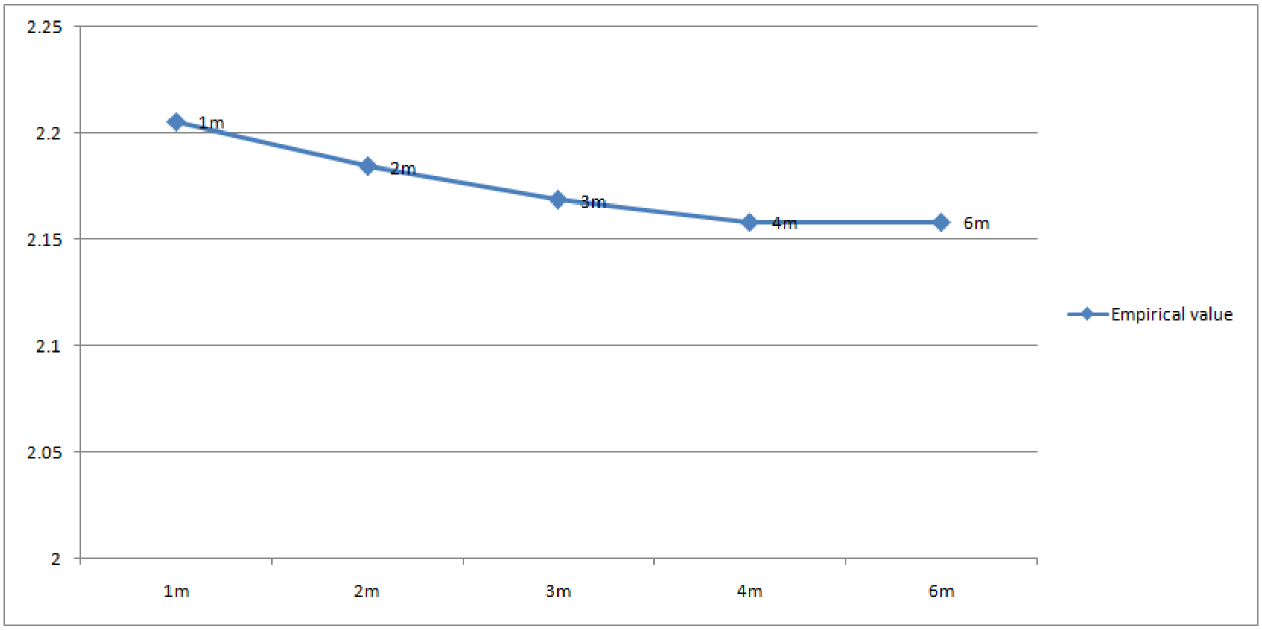


Figure 2. Variance swap rates (expressed in volatility terms as a percentage)

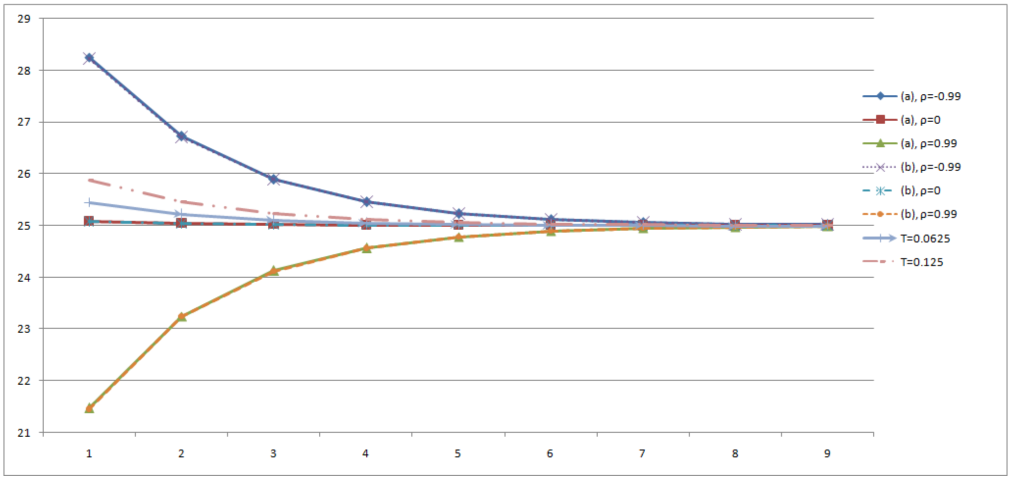


Figure 3. Proportional variance swap rates (expressed in volatility terms as a percentage)

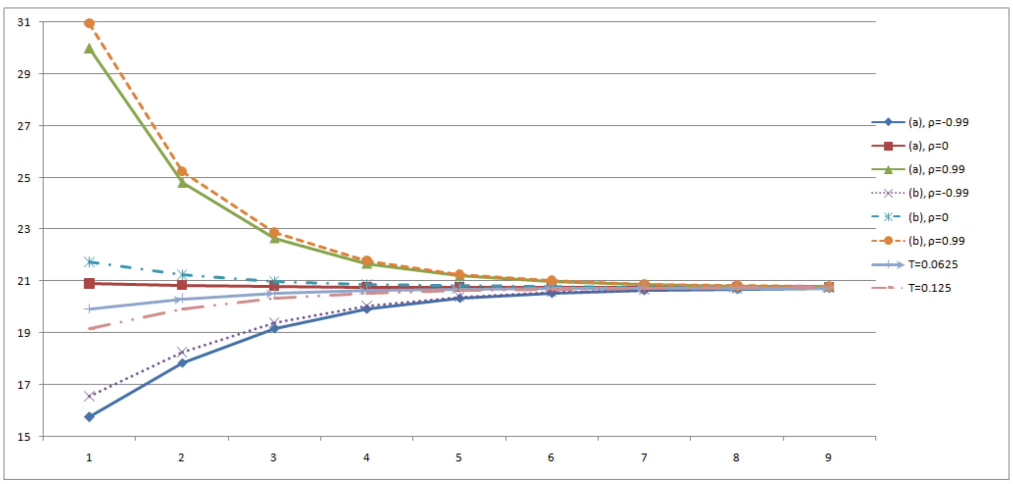
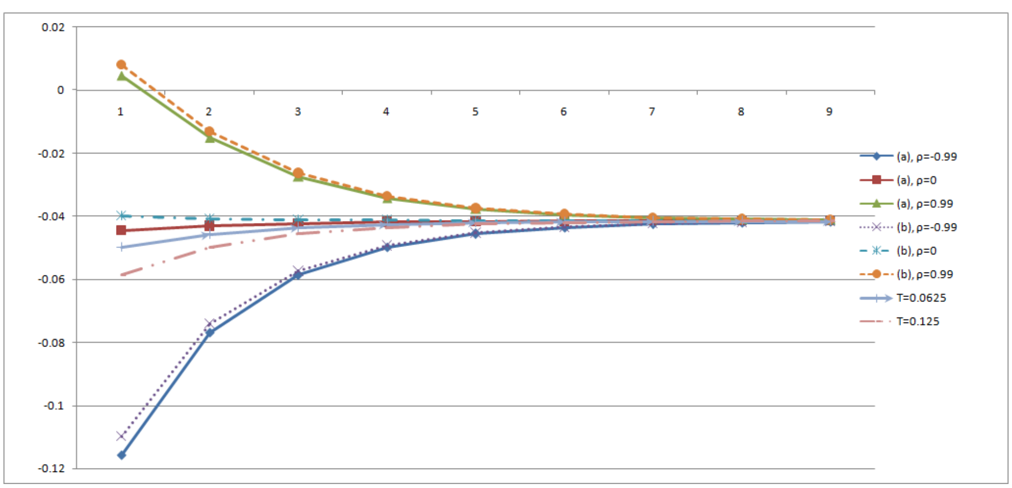


Figure 4. Undiscounted skewness swap prices



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